A

Mud Maps

The wide variety of approaches to and definitions of stability can be confusing. Unfortunately, if one insists on non-countable spaces there is little that can be done about the occasions when two definitions are "almost the same" except to try and delineate the differences.

Here then is an overview of the structure of Markov chains we have developed, at least for the class of chains on which we have concentrated, namely

$$\mathcal{I} := \{\boldsymbol{\Phi} : \boldsymbol{\Phi} \text{ is } \psi\text{-irreducible for some } \psi\}.$$

We have classified chains in $\mathcal{I}$ using three different but (almost) equivalent properties:

P^n-properties : that is, direct properties of the transition laws P^n

τ-properties : properties couched in terms of the hitting times τ_A for appropriate sets A

Drift properties : properties using one step increments of the form of ΔV for some function V.

A.1 Recurrence versus transience

The first fundamental dichotomy (Chapter 8) is

$$\mathcal{I} = \mathcal{T} + \mathcal{R}$$

where $\mathcal{T}$ denotes the class of *transient chains* and $\mathcal{R}$ denotes the class of *recurrent chains.* This is defined as a dichotomy through a P^n-property in Theorem 8.0.1:

$\boldsymbol{P^n}$-Definition of Recurrent and Transient Chains

$$\boldsymbol{\Phi} \in \mathcal{R} \iff \sum_n P^n(x, A) = \infty, \qquad x \in \mathsf{X},\ A \in \mathcal{B}^+(\mathsf{X})$$

$$\boldsymbol{\Phi} \in \mathcal{T} \iff \sum_n P^n(x, A_j) \le M_j < \infty, \qquad x \in \mathsf{X},\ \mathsf{X} = \cup A_j$$

A recurrent chain is "almost" a Harris chain (Chapter 9). Define $\mathcal{H} \subseteq \mathcal{R}$ by the *Harris* τ*-property*

$$\boldsymbol{\Phi} \in \mathcal{H} \iff \mathsf{P}_x(\tau_A < \infty) \equiv 1, \qquad x \in \mathsf{X},\ A \in \mathcal{B}^+(\mathsf{X}).$$

If $\boldsymbol{\Phi} \in \mathcal{R}$ then (Theorem 9.0.1) there is a full absorbing set (a maximal Harris set) H such that

$$\mathsf{X} = H \cup N$$

and $\boldsymbol{\Phi}$ can be restricted in a unique way to a chain $\boldsymbol{\Phi} \in \mathcal{H}$ on the set H.

The τ-classification of $\mathcal{T}$ and $\mathcal{R}$ can be made stronger in terms of

$$Q(x, A) = \mathsf{P}_x(\boldsymbol{\Phi} \in A \text{ i.o.})$$

We have from Theorem 8.0.1 and Theorem 9.0.1:

$\boldsymbol{\tau}$-Classification of Recurrent and Transient Chains

$$\boldsymbol{\Phi} \in \mathcal{R} \iff Q(x, A) = 1, \qquad x \in H,\ A \in \mathcal{B}^+(\mathsf{X})$$

$$\boldsymbol{\Phi} \in \mathcal{T} \iff Q(x, A) = 0, \qquad x \in \mathsf{X},\ A \text{ petite}$$

If indeed $\boldsymbol{\Phi} \in \mathcal{H}$ then the first of these holds for all x since $H = \mathsf{X}$.

The drift classification we have derived is then (Theorem 9.1.8 and Theorem 8.0.2)

Drift Classification of Recurrent and Transient Chains

$$\boldsymbol{\Phi} \in \mathcal{H} \quad \Longleftarrow \quad \begin{array}{l} \Delta V(x) \leq 0, \qquad x \in C^c, \\ C \text{ petite, } V \text{ unbounded off petite sets} \end{array}$$

$$\boldsymbol{\Phi} \in \mathcal{T} \quad \Longleftrightarrow \quad \begin{array}{l} \Delta V(x) \geq 0, \qquad x \in C^c, \\ C \text{ petite, } V \text{ bounded and increasing off } C \end{array}$$

There is thus only one gap in these classifications, namely the actual equivalence of the drift condition for recurrence. We have shown (Theorem 9.4.2) that such equivalence holds for Feller (including countable space) chains.

Finally, it is valuable in practice in a topological context to recall that for T-chains, which (Proposition 6.2.8) include all Feller chains in $\mathcal{I}$ such that $\operatorname{supp}\psi$ has non-empty interior

(i) if $\boldsymbol{\Phi}$ is in $\mathcal{I}$ then (Theorem 6.2.5)

$$\boldsymbol{\Phi} \text{ is a T-chain} \iff \text{every compact set is petite;}$$

(ii) if $\boldsymbol{\Phi}$ is a T-chain in $\mathcal{I}$ then (Theorem 9.2.2)

$$\boldsymbol{\Phi} \in \mathcal{H} \iff \boldsymbol{\Phi} \text{ is non-evanescent;}$$

that is, Harris chains in this case do not leave compact sets forever.

A.2 Positivity versus nullity

The second fundamental dichotomy (Chapter 10) is

$$\mathcal{I} = \mathcal{P} + \mathcal{N}$$

where $\mathcal{N}$ denotes the set of *null chains* and $\mathcal{P} \subseteq \mathcal{R}$ denotes the set of *positive chains*. Since every transient chain is *a fortiori* null, this is in any real sense a breakup of $\mathcal{R}$ rather than the complete set $\mathcal{I}$, and is defined in Chapter 10 through a P^n-property:

First $\boldsymbol{P^n}$-Definition of Positive and Null Chains

$$\boldsymbol{\Phi} \in \mathcal{P} \iff \pi(A) = \int \pi(dy) P^n(y, A), \qquad A \in \mathcal{B}(\mathsf{X})$$

where π is a probability measure with $\pi(\mathsf{X}) = 1$

$$\boldsymbol{\Phi} \in \mathcal{N} \iff \mu(A) \geq \int \mu(dy) P^n(y, A), \qquad A \in \mathcal{B}(\mathsf{X})$$

where μ is a measure with $\mu(\mathsf{X}) = \infty$.

A positive chain is again "almost" a *regular chain*. Define the collection $\mathcal{S} \subseteq \mathcal{P}$ by the *τ-property of regularity*

$$\boldsymbol{\Phi} \in \mathcal{S} \iff \sup_{x \in C_j} \mathsf{E}_x[\tau_A] < \infty, \qquad A \in \mathcal{B}^+(\mathsf{X}),\ \mathsf{X} = \cup C_j.$$

If $\boldsymbol{\Phi} \in \mathcal{P}$ then (Theorem 11.0.1) there is a full absorbing set S such that

$$\mathsf{X} = S \cup N$$

and $\boldsymbol{\Phi}$ can be restricted in a unique way to a regular chain $\boldsymbol{\Phi} \in \mathcal{S}$ on the set S.

The τ-classification of $\mathcal{P}$ and $\mathcal{N}$ can be made stronger, in almost exact analogy to the recurrence classification above. Theorem 11.0.1 shows

$\boldsymbol{\tau}$-Classification of Positive and Null Chains

$$\boldsymbol{\Phi} \in \mathcal{P} \iff \sup_{x \in C_j} \mathsf{E}_x[\tau_A] < \infty, \qquad A \in \mathcal{B}^+(\mathsf{X}),\ S = \cup C_j.$$

$$\boldsymbol{\Phi} \in \mathcal{N} \iff \int_C \pi(dx) \mathsf{E}_x[\tau_C] = \infty, \qquad C \in \mathcal{B}^+(\mathsf{X})$$

Again, if $\boldsymbol{\Phi} \in \mathcal{S}$ then the first of these holds with $S = \mathsf{X}$. We might expect that

$$\boldsymbol{\Phi} \in \mathcal{N} \iff \inf_{x \in C} \mathsf{E}_x[\tau_C] = \infty, \qquad \text{some } C \in \mathcal{B}^+(\mathsf{X}):$$

clearly the infinite expected hitting times will imply the chain is not positive, but the converse appears to be so far unknown except when C is an atom.

The drift classification is

Drift Classification of Positive and Null Chains

$$\boldsymbol{\Phi} \in \mathcal{S} \iff \Delta V(x) \le -1 + b\mathbb{1}_C, \ x \in \mathsf{X}, \ C \text{ petite}$$

$$\boldsymbol{\Phi} \in \mathcal{N} \Longleftarrow \begin{cases} \Delta V(x) \ge 0, & x \in C^c, \\ \int P(x, dy)|V(y) - V(x)| \text{ bounded}, \\ C \text{ petite}, V \text{ increasing off } C. \end{cases}$$

There is again one open question in these classifications, namely that of the equivalence or otherwise of the drift condition for nullity. We do not know how close this is to complete.

In a topological context we know again (see Chapter 18) that for T-chains, there is a further stability property completely equivalent to positivity: if $\boldsymbol{\Phi}$ is an aperiodic T-chain in $\mathcal{R}$ then

$$\boldsymbol{\Phi} \in \mathcal{P} \iff \{P^n(x, \cdot\,)\} \text{ is tight, a.e. } x \in \mathsf{X}.$$

Both the P^n and τ properties are essentially properties involving the whole trajectory of the chain. The drift conditions, and in particular their sufficiency for classification, are powerful practical tools of analysis because they involve only the one-step movement of the chain: this is summarized further in Section B.1.

A.3 Convergence Properties

There is a further P^n-description of $\mathcal{P}$ and $\mathcal{N}$, closer to the recurrence/transience dichotomy, which is developed in Chapter 18, and which is the classical starting point in countable chain theory.

Second $\boldsymbol{P^n}$-Definition of Positive and Null Chains

$$\boldsymbol{\Phi} \in \mathcal{P} \iff \limsup_{n\to\infty} P^n(x, A) > 0, \qquad x \in \mathsf{X},\ A \in \mathcal{B}^+(\mathsf{X})$$

$$\boldsymbol{\Phi} \in \mathcal{N} \iff \lim_{n\to\infty} P^n(x, B_j) = 0, \qquad x \in \mathsf{X},\ \mathsf{X} = \bigcup B_j$$

However, these are weak categorizations of the types of convergence which hold for these chains. For *aperiodic* chains we have (Theorem 13.0.1)

$$\mathcal{H} \cap \mathcal{P} = \mathcal{E}$$

where the class $\mathcal{E}$ is the set of *ergodic chains* such that

$$\boldsymbol{\Phi} \in \mathcal{E} \iff \lim_{n\to\infty} \|P^n(x,\,\cdot\,) - \pi\| = 0, \qquad x \in \mathsf{X}.$$

The properties of $\mathcal{E}$ are delineated further in Part III, and in particular in our next Appendix we summarize criteria (drift conditions) for classifying sub-classes of $\mathcal{E}$.