C

A Glossary of Model Assumptions

Here we gather together the assumptions used for the classes of models we have analyzed as continuing examples. These are only intended for reference. Discussion of the background or the use of these terms is given as they are originally introduced: the Index gives a coherent list of the point of introduction of these assumptions with the nomenclature given here, whilst the equation numbering is that of the original introduction to the model assumption.

C.1 Regenerative Models

We first consider the class of models loosely defined as "regenerative". Such models are usually addressed in applied probability or operations research contexts.

C.1.1 Recurrence time chains

Both discrete time and continuous time renewal processes have served as examples as well as tools in our analysis.

(RT1) If $\{Z_n\}$ is a discrete time renewal process, then the *forward recurrence time chain* $\mathbf{V}^+ = V^+(n), n \in \mathbb{Z}_+$ is given by

$$V^+(n) := \inf(Z_m - n : Z_m > n), \qquad n \geq 0$$

(RT2) The *backward recurrence time chain* $\mathbf{V}^- = V^-(n), n \in \mathbb{Z}_+$ is given by

$$V^-(n) := \inf(n - Z_m : Z_m \leq n), \qquad n \geq 0.$$

(RT3) If $\{Z_n\}$ is a renewal process in continuous time with no delay, then we call the process

$$V^+(t) := \inf(Z_n - t : Z_n > t,\ n \geq 1), \qquad t \geq 0$$

the *forward recurrence time process*; and for any $\delta > 0$, the discrete time chain $\mathbf{V}^+_\delta = V^+(n\delta), n \in \mathbb{Z}_+$ is called the *forward recurrence time δ-skeleton*.

(RT4) We call the process

$$V^-(t) := \inf(t - Z_n : Z_n \leq t,\ n \geq 1), \qquad t \geq 0$$

the *backward recurrence time process*; and for any $\delta > 0$, the discrete time chain $\mathbf{V}^-_\delta = V^-(n\delta), n \in \mathbb{Z}_+$ is called the *backward recurrence time δ-skeleton*.

C.1.2 Random Walk

We have analyzed both random walk on the real line and random walk on the half line, and many models based on these.

(RW1) Suppose that $\boldsymbol{\Phi} = \{\Phi_n; n \in \mathbb{Z}_+\}$ is a collection of random variables defined by choosing an arbitrary distribution for Φ_0 and setting for $k \geq 1$

$$\Phi_k = \Phi_{k-1} + W_k$$

where the W_k are i.i.d. random variables taking values in $\mathbb{R}$ with

$$\Gamma(-\infty, y] = \mathsf{P}(W_n \leq y). \tag{1.6}$$

Then $\boldsymbol{\Phi}$ is called *random walk* on $\mathbb{R}$.

(RW2) We call the random walk spread-out (or equivalently, we call Γ spread out) if some convolution power Γ^{n*} is non-singular with respect to μ^{Leb}.

(RWHL1) Suppose $\boldsymbol{\Phi} = \{\Phi_n; n \in \mathbb{Z}_+\}$ is defined by choosing an arbitrary distribution for Φ_0 and taking

$$\Phi_n = [\Phi_{n-1} + W_n]^+ \tag{1.7}$$

where $[\Phi_{n-1} + W_n]^+ := \max(0, \Phi_{n-1} + W_n)$ and again the W_n are i.i.d. random variables taking values in $\mathbb{R}$ with $\Gamma(-\infty, y] = \mathsf{P}(W \leq y)$.

Then $\boldsymbol{\Phi}$ is called *random walk on a half-line.*

C.1.3 Storage Models and Queues

Random walks provide the underlying structure for both queueing and storage models, and we have assumed several specializations for these physical systems.

Queueing models and storage models are closely related in formal structure, although the physical interpretation of the quantities of interest are somewhat different.

We have analyzed GI/G/1 queueing models under the assumptions

(Q1) Customers arrive into a service operation at timepoints $T_0 = 0$, $T_0 + T_1$, $T_0 + T_1 + T_2, \ldots$ where the interarrival times T_i, $i \geq 1$, are i.i.d. random variables, distributed as a random variable T with $G(-\infty, t] = \mathsf{P}(T \leq t)$.

(Q2) The n^{th} customer brings a job requiring service S_n where the service times are independent of each other and of the interarrival times, and are distributed as a variable S with distribution $H(-\infty, t] = \mathsf{P}(S \leq t)$.

(Q3) There is one server and customers are served in order of arrival.

In such a general situation we have often considered the countable space chain consisting of the number of customers in the queue either at arrival or at departure times. Under some exponential assumptions these give the GI/M/1 and M/G/1 queueing systems:

(Q4) If the distribution $H(-\infty, t]$ of service times is exponential with

$$H(-\infty, t] = 1 - e^{-\mu t}, \quad t \geq 0$$

then the queue is called a GI/M/1 queue.

(Q5) If the distribution $G(-\infty, t]$ of inter-arrival times is exponential with

$$G(-\infty, t] = 1 - e^{-\lambda t}, \quad t \geq 0$$

then the queue is called an M/G/1 queue.

In storage models we have a special case of random walk on a half line, but here we consider the model at the times of input and break the increment into the input and output components.

The *simple storage model* has the assumptions

(SSM1) For each $n \geq 0$ let S_n and T_n be i.i.d. random variables on $\mathbb{R}$ with distributions H and G.

(SSM2) Define the random variables

$$\Phi_{n+1} = [\Phi_n + S_n - J_n]^+$$

where the variables J_n are i.i.d., with

$$\mathsf{P}(J_n \leq x) = G(-\infty, x/r] \tag{2.32}$$

for some $r > 0$.

Then the chain $\boldsymbol{\Phi} = \{\Phi_n\}$ represents the contents of a storage system at the times $\{T_n-\}$ immediately before each input, and is called the *simple storage model*, with release rate r.

More complex content-dependent storage models have the assumptions

(CSM1) For each $n \geq 0$ let $S_n(x)$ and T_n be i.i.d. random variables on $\mathbb{R}$ with distributions H_x and G.

(CSM2) Define the random variables

$$\Phi_{n+1} = [\Phi_n - J_n + S_n(\Phi_n - J_n)]^+$$

where the variables J_n are independently distributed, with

$$\mathsf{P}(J_n \leq y \mid \Phi_n = x) = \int G(dt)\mathsf{P}(J_x(t) \leq y) \tag{2.34}$$

The chain $\boldsymbol{\Phi} = \{\Phi_n\}$ can be interpreted as the content of the storage system at the times $\{T_n-\}$ immediately before each input, and is called the *content dependent storage model.*

We also note that these models can be used to represent a number of state-dependent queueing systems where the rate of service depends on the actual state of the system rather than being independent.

C.2 State Space Models

The other broad class of models we have considered are loosely described as "state space models", and occur in communication and control engineering, other areas of systems analysis, and in time series.

C.2.1 Linear Models

The process $\mathbf{X} = \{X_n , n \in \mathbb{Z}_+\}$ is called the *simple linear model* if

(SLM1) X_n and W_n are random variables on $\mathbb{R}$ satisfying, for some $\alpha \in \mathbb{R}$,

$$X_n = \alpha X_{n-1} + W_n, \qquad n \geq 1;$$

(SLM2) the random variables $\{W_n\}$ are an i.i.d. sequence with distribution Γ on $\mathbb{R}$.

Next suppose $\mathbf{X} = \{X_k\}$ is a stochastic process for which

(LSS1) There exists an $n \times n$ matrix F and an $n \times p$ matrix G such that for each $k \in \mathbb{Z}_+$, the random variables X_k and W_k take values in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, and satisfy inductively for $k \geq 1$, and arbitrary W_0,

$$X_k = FX_{k-1} + GW_k;$$

(LSS2) The random variables $\{W_n\}$ are i.i.d. with common finite mean, taking values on $\mathbb{R}^p$, with distribution $\Gamma(A) = \mathsf{P}(W_j \in A)$.

Then $\mathbf{X}$ is called the *linear state space model driven by* F, G, or the LSS(F,G) model, with associated control model LCM(F,G) (defined below).

Further assumptions are required for the stability analysis of this model. These include, at different times

(LSS3) The noise variable W has a Gaussian distribution on $\mathbb{R}^p$ with zero mean and unit variance: that is, $W \sim N(0, I)$.

(LSS4) The distribution Γ of the random variable W is non-singular with respect to Lebesgue measure, with non-trivial density γ_w.

(LSS5) The eigenvalues of F fall within the open unit disk in $\mathbb{C}$.

The *associated (linear) control model* LCM(F,G) is defined by the following two sets of assumptions.

Suppose $\mathbf{x} = \{x_k\}$ is a deterministic process on $\mathbb{R}^n$ and $\mathbf{u} = \{u_n\}$ is a deterministic process on $\mathbb{R}^p$, for which x_0 is arbitrary; then $\mathbf{x}$ is called the *linear control model driven by* F, G, or the LCM(F,G) model, if for $k \geq 1$

(LCM1) there exists an $n \times n$ matrix F and an $n \times p$ matrix G such that for each $k \in \mathbb{Z}_+$,

$$x_{k+1} = Fx_k + Gu_{k+1}; \qquad (1.4)$$

(LCM2) the sequence $\{u_n\}$ on $\mathbb{R}^p$ is chosen deterministically.

A process $\mathbf{Y} = \{Y_n\}$ is called a (scalar) *autoregression of order* k, or AR(k) model, if it satisfies

(AR1) for each $n \geq 0$, Y_n and W_n are random variables on $\mathbb{R}$, satisfying, inductively for $n \geq k$,
$$Y_n = \alpha_1 Y_{n-1} + \alpha_2 Y_{n-2} + \ldots + \alpha_k Y_{n-k} + W_n,$$
for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$;

(AR2) the sequence $\mathbf{W}$ is an error or innovation sequence on $\mathbb{R}$.

The process $\mathbf{Y} = \{Y_n\}$ is called an *autoregressive-moving average process of order* (k, ℓ), or ARMA(k,ℓ) model, if it satisfies

(ARMA1) for each $n \geq 0$, Y_n and W_n are random variables on $\mathbb{R}$, satisfying, inductively for $n \geq k$,
$$Y_n = \sum_{j=1}^{k} \alpha_j Y_{n-j} + \sum_{j=1}^{\ell} \beta_j W_{n-j} + W_n,$$
for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell \in \mathbb{R}$;

(ARMA2) the sequence $\mathbf{W}$ is an error or innovation sequence on $\mathbb{R}$.

C.2.2 Nonlinear Models

The stochastic nonlinear systems we analyze have a deterministic analogue in *semi-dynamical systems*, defined by:

(DS1) The process $\boldsymbol{\Phi}$ is deterministic, and generated by the nonlinear difference equation, or semi-dynamical system,
$$\Phi_{k+1} = F(\Phi_k), \qquad k \in \mathbb{Z}_+, \tag{11.16}$$
where $F \colon \mathsf{X} \to \mathsf{X}$ is a continuous function.

(DS2) There exists a positive function $V \colon \mathsf{X} \to \mathbb{R}_+$ and a compact set $C \subset \mathsf{X}$ and constant $M < \infty$ such that
$$\Delta V(x) := V(F(x)) - V(x) \leq -1$$
for all x lying outside the compact set C, and
$$\sup_{x \in C} V(F(x)) \leq M.$$

The chain $\mathbf{X} = \{X_n\}$ is called a *scalar nonlinear state space model on* $\mathbb{R}$ *driven by* F, or SNSS(F) model, if it satisfies

(SNSS1) for each $n \geq 0$, X_n and W_n are random variables on $\mathbb{R}$, satisfying, inductively for $n \geq 1$,
$$X_n = F(X_{n-1}, W_n),$$
for some smooth (C^∞) function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

We also use, for various results at various times,

(SNSS2) The sequence $\mathbf{W}$ is a disturbance sequence on $\mathbb{R}$, whose marginal distribution Γ possesses a density γ_w supported on an open set O_w, called the *control set.*

(SNSS3) The distribution Γ of W is absolutely continuous, with a density γ_w on $\mathbb{R}$ which is lower semicontinuous.

Suppose $\mathbf{X} = \{X_k\}$, where

(NSS1) for each $k \geq 0$, X_k and W_k are random variables on $\mathbb{R}^n$, $\mathbb{R}^p$ respectively, satisfying inductively for $k \geq 1$,

$$X_k = F(X_{k-1}, W_k),$$

for some smooth (C^∞) function $F\colon \mathsf{X} \times O_w \to \mathsf{X}$, where X is an open subset of $\mathbb{R}^n$, and O_w is an open subset of $\mathbb{R}^p$.

Then $\mathbf{X}$ is called a *nonlinear state space model driven by F, or NSS(F) model, with control set O_w.*

Again for various properties to hold we require

(NSS2) The random variables $\{W_k\}$ are a disturbance sequence on $\mathbb{R}^p$, whose marginal distribution Γ possesses a density γ_w which is supported on an open set O_w.

(NSS3) The distribution Γ of W possesses a density γ_w on $\mathbb{R}^p$ which is lower semicontinuous, and the *control set* is the open set

$$O_w := \{x \in \mathbb{R} : \gamma_w(x) > 0\}.$$

The *associated control model* CM(F) is defined as follows.

(CM1) The deterministic system

$$x_k = F_k(x_0, u_1, \dots, u_k), \qquad k \in \mathbb{Z}_+, \tag{2.8}$$

where the sequence of maps $\{F_k : \mathsf{X} \times O_w^k \to \mathsf{X} : k \geq 0\}$ is defined by (2.5), is called the associated control system for the NSS(F) model (denoted CM(F)) provided the deterministic control sequence $\{u_1, \dots, u_k, k \in \mathbb{Z}_+\}$ lies in the control set $O_w \subseteq \mathbb{R}^p$.

To obtain a T-chain, we assume for the SNSS(F) model,

(CM2) For each initial condition $x_0^0 \in \mathbb{R}$ there exists $k \in \mathbb{Z}_+$ and a sequence $(u_1^0, \dots, u_k^0) \in O_w^k$ such that the derivative

$$\Big[\frac{\partial}{\partial u_1} F_k\left(x_0^0, u_1^0, \dots, u_k^0\right) \mid \cdots \mid \frac{\partial}{\partial u_k} F_k\left(x_0^0, u_1^0, \dots, u_k^0\right)\Big] \tag{7.4}$$

is non-zero.

For the multi-dimensional NSS(F) model we often assume

(CM3) For each initial condition $x_0^0 \in \mathbb{R}$ there exists $k \in \mathbb{Z}_+$ and a sequence $\vec{u}^0 = (u_1^0, \ldots, u_k^0) \in O_w^k$ such that

$$\operatorname{rank} C_x^k(\vec{u}^0) = n. \tag{7.13}$$

A specific example of the NSS(F) model is the nonlinear autoregressive-moving average, or NARMA, model.

The process $\mathbf{Y} = \{Y_n\}$ is called a *nonlinear autoregressive-moving average process of order* (k, ℓ) if the values $Y_0, \ldots, Y_{k-1}$ are arbitrary and

(NARMA1) for each $n \geq 0$, Y_n and W_n are random variables on $\mathbb{R}$, satisfying, inductively for $n \geq k$,

$$Y_n = G(Y_{n-1}, Y_{n-2}, \ldots, Y_{n-k}, W_n, W_{n-1}, W_{n-2}, \ldots, W_{n-\ell})$$

where the function $G: \mathbb{R}^{k+\ell+1} \to \mathbb{R}$ is smooth (C^∞).

(NARMA2) the sequence $\mathbf{W}$ is an error sequence on $\mathbb{R}$.

C.2.3 Particular Examples

The *simple adaptive control model* is a triple $\mathbf{Y}, \mathbf{U}, \boldsymbol{\theta}$ where

(SAC1) the output sequence $\mathbf{Y}$ and parameter sequence $\boldsymbol{\theta}$ are defined inductively for any input sequence $\mathbf{U}$ by

$$Y_{k+1} = \theta_k Y_k + U_k + W_{k+1} \tag{2.19}$$

$$\theta_{k+1} = \alpha\theta_k + Z_{k+1}, \qquad k \geq 1 \tag{2.20}$$

where α is a scalar with $|\alpha| < 1$;

(SAC2) the bivariate disturbance process $\binom{\mathbf{Z}}{\mathbf{W}}$ is Gaussian and satisfies

$$\begin{aligned} \mathsf{E}[\tbinom{Z_n}{W_n}] &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mathsf{E}[\tbinom{Z_n}{W_n}(Z_k, W_k)] &= \begin{pmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix} \delta_{n-k}, \qquad n \geq 1 \end{aligned}$$

with $\sigma_z < 1$;

(SAC3) the input process satisfies $U_k \in \mathcal{Y}_k$, $k \in \mathbb{Z}_+$, where $\mathcal{Y}_k = \sigma\{Y_0, \ldots, Y_k\}$.

With the control U_k chosen as $U_k = -\hat{\theta}_k Y_k$, $k \in \mathbb{Z}_+$, the closed loop system equations for the simple adaptive control model are

$$\tilde{\theta}_{k+1} = \alpha\tilde{\theta}_k - \alpha\Sigma_k Y_{k+1} Y_k (\Sigma_k Y_k^2 + \sigma_w^2)^{-1} + Z_{k+1} \tag{2.21}$$

$$Y_{k+1} = \tilde{\theta}_k Y_k + W_{k+1} \tag{2.22}$$

$$\Sigma_{k+1} = \sigma_z^2 + \alpha^2\sigma_w^2 \Sigma_k (\Sigma_k Y_k^2 + \sigma_w^2)^{-1}, \qquad k \geq 1 \tag{2.23}$$

where the triple $\Sigma_0, \tilde{\theta}_0, Y_0$ is given as an initial condition.

The closed loop system gives rise to a Markovian system of the form (NSS1), so that $\Phi_k = (\Sigma_k, \tilde{\theta}_k, Y_k)^\top$ is a Markov chain with state space $\mathsf{X} = [\sigma_z^2, \frac{\sigma_z^2}{1-\alpha^2}] \times \mathbb{R}^2$.

A chain $\mathbf{X} = \{X_n\}$ is called a *scalar self-exciting threshold autoregression (or SETAR) model* if it satisfies

(SETAR1) for each $1 \le j \le M$, X_n and $W_n(j)$ are random variables on $\mathbb{R}$, satisfying, inductively for $n \ge 1$,

$$X_n = \phi(j) + \theta(j)X_{n-1} + W_n(j), \qquad r_{j-1} < X_{n-1} \le r_j,$$

where $-\infty = r_0 < r_1 < \cdots < r_M = \infty$ and $\{W_n(j)\}$ forms an i.i.d. zero-mean error sequence for each j, independent of $\{W_n(i)\}$ for $i \ne j$.

For stability classification we often use

(SETAR2) For each $j = 1, \cdots, M$, the noise variable $W(j)$ has a density positive on the whole real line.

(SETAR3) The variances of the noise distributions for the two end intervals are finite; that is,

$$\mathsf{E}(W^2(1)) < \infty, \qquad \mathsf{E}(W^2(M)) < \infty$$

A chain $\mathbf{X} = \{X_n\}$ is called a *simple (first order) bilinear process* if it satisfies

(SBL1) For each $n \ge 0$, X_n and W_n are random variables on $\mathbb{R}$, satisfying for $n \ge 1$,

$$X_n = \theta X_{n-1} + bX_{n-1}W_n + W_n$$

where θ and b are scalars, and the sequence $\mathbf{W}$ is an error sequence on $\mathbb{R}$.

(SBL2) The sequence $\mathbf{W}$ is a disturbance process on $\mathbb{R}$, whose marginal distribution Γ possesses a finite second moment, and a density γ_w which is lower semicontinuous.

The process $\boldsymbol{\Phi} = \binom{\boldsymbol{\theta}}{\mathbf{Y}}$ is called the *dependent parameter bilinear model* if it satisfies

(DBL1) For some $|\alpha| < 1$ and all $k \in \mathbb{Z}_+$,

$$Y_{k+1} = \theta_k Y_k + W_{k+1} \tag{2.12}$$

$$\theta_{k+1} = \alpha\theta_k + Z_{k+1}. \tag{2.13}$$

We often also require

(DBL2) The joint process $(\mathbf{Z}, \mathbf{W})^\top$ is a disturbance sequence on $\mathbb{R}^2$, $\mathbf{Z}$ and $\mathbf{W}$ are mutually independent, and the distributions Γ_w and Γ_z of W, Z respectively possess densities which are lower semicontinuous. It is assumed that W has a finite second moment, and that $\mathsf{E}[\log(1 + |Z|)] < \infty$.

The chain $\mathbf{X} = \{X_k\}$ is called a *random coefficient autoregression (RCA) process* if it satisfies, for each $k \ge 0$,

$$X_{k+1} = (A + \Gamma_{k+1})X_k + W_{k+1}$$

where X_k, Γ_k and W_k are random variables satisfying the following:

(RCA1) The sequences $\boldsymbol{\Gamma}$ and $\mathbf{W}$ are i.i.d. and also independent of each other.

Conditions which lead to stability are then

(RCA2) The following expectations exist, and have the prescribed values:

$$\begin{array}{llll} \mathsf{E}[W_k] = 0 & & \mathsf{E}[W_k W_k^\top] = G & (n \times n), \\ \mathsf{E}[\Gamma_k] = 0 & (n \times n) & \mathsf{E}[\Gamma_k \otimes \Gamma_k] = C & (n^2 \times n^2), \end{array}$$

and the eigenvalues of $A \otimes A + C$ have moduli less than unity.

(RCA3) The distribution of $\binom{\Gamma_k}{W_k}$ has an everywhere positive density with respect to μ^{Leb} on $\mathbb{R}^{n^2+p}$