14

f-Ergodicity and f-Regularity

In Chapter 13 we considered ergodic chains for which the limit

$$\lim_{k\to\infty} \mathsf{E}_x[f(\Phi_k)] = \int f\, d\pi \tag{14.1}$$

exists for every initial condition, and every bounded function f on X.

An assumption that f is bounded is often unsatisfactory in applications. For example, f may denote a cost function in an optimal control problem, in which case $f(\Phi_n)$ will typically be a norm-like function of Φ_n on X; in queueing applications, the function $f(x)$ might denote buffer levels in a queue corresponding to the particular state $x \in \mathsf{X}$ which is, again, typically an unbounded function on X; in storage models, f may denote penalties for high values of the storage level, which correspond to overflow penalties in reality.

The purpose of this chapter is to relax the boundedness condition by developing more general formulations of regularity and ergodicity. Our aim is to obtain convergence results of the form (14.1) for the mean value of $f(\Phi_k)$, where $f\colon \mathsf{X} \to [1,\infty)$ is an arbitrary fixed function. As in Chapter 13, it will be shown that the simplest approach to ergodic theorems of this kind is to consider simultaneously all functions which are dominated by f: that is, to consider convergence in the f-norm, defined as

$$\|\nu\|_f = \sup_{g:|g|\le f} |\nu(g)|$$

where ν is any signed measure.

The goals described above are achieved in the following f-Norm Ergodic Theorem for aperiodic chains.

Theorem 14.0.1 (f-Norm Ergodic Theorem) *Suppose that the chain $\boldsymbol{\Phi}$ is ψ-irreducible and aperiodic, and let $f \ge 1$ be a function on X. Then the following conditions are equivalent:*

(i) *The chain is positive recurrent with invariant probability measure π and*

$$\pi(f) := \int \pi(dx) f(x) < \infty$$

(ii) *There exists some petite set $C \in \mathcal{B}(\mathsf{X})$ such that*

$$\sup_{x\in C} \mathsf{E}_x\Big[\sum_{n=0}^{\tau_C-1} f(\Phi_n)\Big] < \infty. \tag{14.2}$$

(iii) *There exists some petite set C and some extended-valued non-negative function V satisfying $V(x_0) < \infty$ for some $x_0 \in \mathsf{X}$, and*

$$\Delta V(x) \leq -f(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{14.3}$$

Any of these three conditions imply that the set $S_V = \{x : V(x) < \infty\}$ is absorbing and full, where V is any solution to (14.3) satisfying the conditions of (iii), and any sublevel set of V satisfies (14.2); and for any $x \in S_V$,

$$\|P^n(x,\,\cdot\,) - \pi\|_f \to 0 \tag{14.4}$$

as $n \to \infty$. Moreover, if $\pi(V) < \infty$ then there exists a finite constant B_f such that for all $x \in S_V$,

$$\sum_{n=0}^{\infty} \|P^n(x,\,\cdot\,) - \pi\|_f \leq B_f(V(x) + 1). \tag{14.5}$$

PROOF The equivalence of (i) and (ii) follows from Theorem 14.1.1 and Theorem 14.2.11. The equivalence of (ii) and (iii) is in Theorems 14.2.3 and 14.2.4, and the fact that sublevel sets of V are "self-regular" as in (14.2) is shown in Theorem 14.2.3. The limit theorems are then contained in Theorems 14.3.3, 14.3.4 and 14.3.5. □

Much of this chapter is devoted to proving this result, and related f-regularity properties which follow from (14.2), and the pattern is not dissimilar to that in the previous chapter: indeed, those ergodicity results, and the equivalences in Theorem 13.0.1, can be viewed as special cases of the general f results we now develop.

The f-norm limit (14.4) obviously implies that the simpler limit (14.1) also holds. In fact, if g is any function satisfying $|g| \leq c(f+1)$ for some $c < \infty$ then $\mathsf{E}_x[g(\Phi_k)] \to \int g\,d\pi$ for states x with $V(x) < \infty$, for V satisfying (14.3). We formalize the behavior we will analyze in

f-Ergodicity

We shall say that the Markov chain $\boldsymbol{\Phi}$ is *f-ergodic* if $f \geq 1$ and

(i) $\boldsymbol{\Phi}$ is positive Harris recurrent with invariant probability π

(ii) the expectation $\pi(f)$ is finite

(iii) for every initial condition of the chain,

$$\lim_{k\to\infty} \|P^k(x,\,\cdot\,) - \pi\|_f = 0.$$

The f-Norm Ergodic Theorem states that if any one of the equivalent conditions of the Aperiodic Ergodic Theorem holds then the simple additional condition that $\pi(f)$ is finite is enough to ensure that a full absorbing set exists on which the chain is f-ergodic. Typically the way in which finiteness of $\pi(f)$ would be established in an application is through finding a test function V satisfying (14.3): and if, as will typically happen, V is finite everywhere then it follows that the chain is f-ergodic without restriction, since then $S_V = \mathsf{X}$.

14.1 f-Properties: chains with atoms

14.1.1 f-Regularity for chains with atoms

We have already given the pattern of approach in detail in Chapter 13. It is not worthwhile treating the countable case completely separately again: as was the case for ergodicity properties, a single accessible atom is all that is needed, and we will initially develop f-ergodic theorems for chains possessing such an atom.

The generalization from total variation convergence to f-norm convergence given an initial accessible atom $\boldsymbol{\alpha}$ can be carried out based on the developments of Chapter 13, and these also guide us in developing characterizations of the initial measures λ for which general f-ergodicity might be expected to hold. It is in this part of the analysis, which corresponds to bounding the first term in the Regenerative Decomposition of Theorem 13.2.5, that the hard work is needed, as we now discuss.

Suppose that $\boldsymbol{\Phi}$ admits an atom $\boldsymbol{\alpha}$ and is positive Harris recurrent with invariant probability measure π. Let $f \geq 1$ be arbitrary: that is, we place no restrictions on the boundedness or otherwise of f. Recall that for any probability measure λ we have from the Regenerative Decomposition that for arbitrary $|g| \leq f$,

$$|\mathsf{E}_\lambda[g(\Phi_n)] - \pi(g)| \;\leq\; \int \lambda(dx) \int {}_\alpha P^n(x, dw) f(w) + |\, a_\lambda * u - \pi(\boldsymbol{\alpha})\,| * t_f\,(n) + \pi(\boldsymbol{\alpha}) \sum_{j=n+1}^{\infty} t_f(j). \tag{14.6}$$

Using hitting time notation we have

$$\sum_{n=1}^{\infty} t_f(n) \;=\; \mathsf{E}_\alpha\Big[\sum_{j=1}^{\tau_\alpha} f(\Phi_j)\Big] \tag{14.7}$$

and thus the finiteness of this expectation will guarantee convergence of the third term in (14.6), as it did in the case of the ergodic theorems in Chapter 13. Also as in Chapter 13, the central term in (14.6) is controlled by the convergence of the renewal sequence u regardless of f, provided the expression in (14.7) is finite.

Thus it is only the first term in (14.6) that requires a condition other than ergodicity and finiteness of (14.7). Somewhat surprisingly, for unbounded f this is a much more troublesome term to control than for bounded f, when it is a simple consequence of recurrence that it tends to zero. This first term can be expressed alternatively as

$$\int \lambda(dx) \int {}_\alpha P^n(x, dw) f(w) = \mathsf{E}_\lambda[f(\Phi_n)\mathbb{1}(\tau_\alpha \geq n)] \tag{14.8}$$

and so we have the representation

$$\sum_{n=1}^{\infty} \int \lambda(dx) \int {}_{\alpha}P^n(x, dw) f(w) \quad = \quad \mathsf{E}_\lambda\Big[\sum_{j=1}^{\tau_\alpha} f(\Phi_j)\Big]. \tag{14.9}$$

This is similar in form to (14.7), and if (14.9) is finite, then we have the desired conclusion that (14.8) does tend to zero. In fact, it is only the sum of these terms that appears tractable, and for this reason it is in some ways more natural to consider the summed form (14.5) rather than simple f-norm convergence.

Given this motivation to require finiteness of (14.7) and (14.9), we introduce the concept of *f-regularity* which strengthens our definition of ordinary regularity.

f-Regularity

A set $C \in \mathcal{B}(\mathsf{X})$ is called *f-regular* where $f\colon \mathsf{X} \to [1, \infty)$ is a measurable function, if for each $B \in \mathcal{B}^+(\mathsf{X})$,

$$\sup_{x \in C} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} f(\Phi_k)\Big] < \infty.$$

A measure λ is called *f-regular* if for each $B \in \mathcal{B}^+(\mathsf{X})$,

$$\mathsf{E}_\lambda\Big[\sum_{k=0}^{\tau_B - 1} f(\Phi_k)\Big] < \infty.$$

The chain $\boldsymbol{\Phi}$ is called *f-regular* if there is a countable cover of X with f-regular sets.

From this definition an f-regular state, seen as a singleton set, is a state x for which $\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} f(\Phi_k)\Big] < \infty$, $B \in \mathcal{B}^+(\mathsf{X})$.

As with regularity, this definition of f-regularity appears initially to be stronger than required since it involves all sets in $\mathcal{B}^+(\mathsf{X})$; but we will show this to be again illusory.

A first consequence of f-regularity, and indeed of the weaker "self-f-regular" form in (14.2), is

Proposition 14.1.1 *If $\boldsymbol{\Phi}$ is recurrent with invariant measure π and there exists $C \in \mathcal{B}(\mathsf{X})$ satisfying $\pi(C) < \infty$ and*

$$\sup_{x \in C} \mathsf{E}_x[\sum_{n=0}^{\tau_C - 1} f(\Phi_n)] < \infty \tag{14.10}$$

then $\boldsymbol{\Phi}$ is positive recurrent and $\pi(f) < \infty$.

PROOF First of all, observe that under (14.10) the set C is Harris recurrent and hence $C \in \mathcal{B}^+(\mathsf{X})$ by Proposition 9.1.1. The invariant measure π then satisfies, from Theorem 10.4.9,

$$\pi(f) = \int_C \pi(dy)\mathsf{E}_y\Big[\sum_{n=0}^{\tau_C - 1} f(\Phi_n)\Big].$$

If C satisfies (14.10) then the expectation is uniformly bounded on C itself, so that $\pi(f) \le \pi(C) M_C < \infty$. □

Although f-regularity is a requirement on the hitting times of all sets, when the chain admits an atom it reduces to a requirement on the hitting times of the atom as was the case with regularity.

Proposition 14.1.2 *Suppose $\boldsymbol{\Phi}$ is positive recurrent with $\pi(f) < \infty$, and that an atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$ exists.*

(i) *Any set $C \in \mathcal{B}(\mathsf{X})$ is f-regular if and only if*

$$\sup_{x \in C} \mathsf{E}_x\Big[\sum_{k=0}^{\sigma_\alpha} f(\Phi_k)\Big] < \infty.$$

(ii) *There exists an increasing sequence of sets $S_f(n)$ where each $S_f(n)$ is f-regular and the set $S_f = \cup S_f(n)$ is full and absorbing.*

PROOF Consider the function $G_\alpha(x, f)$ previously defined in (11.21) by

$$G_\alpha(x, f) = \mathsf{E}_x[\sum_{k=0}^{\sigma_\alpha} f(\Phi_k)]. \tag{14.11}$$

When $\pi(f) < \infty$, by Theorem 11.3.5 the bound $PG_\alpha(x, f) \le G_\alpha(x, f) + c$ holds for the constant $c = \mathsf{E}_\alpha[\sum_{k=1}^{\tau_\alpha} f(\Phi_k)] = \pi(f)/\pi(\boldsymbol{\alpha}) < \infty$, which shows that the set $\{x : G_\alpha(x, f) < \infty\}$ is absorbing, and hence by Proposition 4.2.3 this set is full.

To prove (i), let B be any sublevel set of the function $G_\alpha(x, f)$ with $\pi(B) > 0$ and apply the bound

$$G_\alpha(x, f) \le \mathsf{E}_x[\sum_{k=0}^{\tau_B - 1} f(\Phi_k)] + \sup_{y \in B} \mathsf{E}_y[\sum_{k=0}^{\sigma_\alpha} f(\Phi_k)].$$

This shows that $G_\alpha(x, f)$ is bounded on C if C is f-regular, and proves the "only if" part of (i).

We have from Theorem 10.4.9 that for any $B \in \mathcal{B}^+(\mathsf{X})$,

$$\begin{aligned}
\infty &> \int_B \pi(dx)\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B} f(\Phi_k)\Big] \\
&\ge \int_B \pi(dx)\mathsf{E}_x\Big[\mathbb{1}(\sigma_\alpha < \tau_B) \sum_{k=\sigma_\alpha+1}^{\tau_B} f(\Phi_k)\Big] \\
&= \int_B \pi(dx)\mathsf{P}_x(\sigma_\alpha < \tau_B)\mathsf{E}_\alpha\Big[\sum_{k=1}^{\tau_B} f(\Phi_k)\Big]
\end{aligned}$$

where to obtain the last equality we have conditioned at time σ_α and used the strong Markov property.

Since $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$ we have that

$$\pi(\boldsymbol{\alpha}) = \int_B \pi(dx)\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \mathbb{1}(\Phi_k \in \boldsymbol{\alpha})\Big] > 0,$$

which shows that $\int_B \pi(dx)\mathsf{P}_x(\sigma_\alpha < \tau_B) > 0$. Hence from the previous bounds, $\mathsf{E}_\alpha\Big[\sum_{k=1}^{\tau_B} f(\Phi_k)\Big] < \infty$ for $B \in \mathcal{B}^+(\mathsf{X})$.

Using the bound $\tau_B \leq \sigma_\alpha + \theta^{\sigma_\alpha}\tau_B$, we have for arbitrary $x \in \mathsf{X}$,

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B} f(\Phi_k)\Big] \leq \mathsf{E}_x\Big[\sum_{k=0}^{\sigma_\alpha} f(\Phi_k)\Big] + \mathsf{E}_\alpha\Big[\sum_{k=1}^{\tau_B} f(\Phi_k)\Big] \tag{14.12}$$

and hence C is f-regular if $G_\alpha(x,f)$ is bounded on C, which proves (i).

To prove (ii), observe that from (14.12) we have that the set $S_f(n) := \{x : G_\alpha(x,f) \leq n\}$ is f-regular, and so the proposition is proved. □

14.1.2 f-Ergodicity for chains with atoms

As we have foreshadowed, f-regularity is exactly the condition needed to obtain convergence in the f-norm.

Theorem 14.1.3 *Suppose that $\boldsymbol{\Phi}$ is positive Harris, aperiodic, and that an atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$ exists.*

(i) *If $\pi(f) < \infty$ then the set S_f of f-regular states is absorbing and full, and for any $x \in S_f$ we have*
$$\|P^k(x,\,\cdot\,) - \pi\|_f \to 0, \qquad k \to \infty.$$

(ii) *If $\boldsymbol{\Phi}$ is f-regular then $\boldsymbol{\Phi}$ is f-ergodic.*

(iii) *There exists a constant $M_f < \infty$ such that for any two f-regular initial distributions λ and μ,*

$$\begin{aligned}\sum_{n=1}^{\infty} \int\!\!\int \lambda(dx)\mu(dy)\|P^n(x,\,\cdot\,) - P^n(y,\,\cdot\,)\|_f \\ \leq \; M_f\Big(\int \lambda(dx)G_\alpha(x,f) + \int \mu(dy)G_\alpha(y,f)\Big).\end{aligned} \tag{14.13}$$

PROOF From Proposition 14.1.2 (ii), the set of f-regular states S_f is absorbing and full when $\pi(f) < \infty$. If we can prove $\|P^k(x,\,\cdot\,) - \pi\|_f \to 0$, for $x \in S_f$, this will establish both (i) and (ii).

But this f-norm convergence follows from (14.6), where the first term tends to zero since x is f-regular, so that $\mathsf{E}_x[\sum_{n=1}^{\tau_\alpha} f(\Phi_n)] < \infty$; the third term tends to zero since $\sum_{n=1}^{\infty} t_f(j) = \mathsf{E}_\alpha[\sum_{n=1}^{\tau_\alpha} f(\Phi_n)] = \pi(f)/\pi(\boldsymbol{\alpha}) < \infty$; and the central term converges to zero by Lemma D.7.1 and the fact that $\boldsymbol{\alpha}$ is an ergodic atom.

To prove the result in (iii), we use the same method of proof as for the ergodic case. By the triangle inequality it suffices to assume that one of the initial distributions is δ_α. We again use the first form of the Regenerative Decomposition to see that for any $|g| \leq f$, $x \in \mathsf{X}$, the sum

$$\sum_{n=1}^{\infty} \int \lambda(dx) |P^n(x,g) - P^n(\boldsymbol{\alpha},g)|$$

is bounded by the sum of the following two terms:

$$\sum_{n=1}^{\infty} \int \lambda(dx) {}_{\alpha}P^n(x,f) \quad = \quad \mathsf{E}_\lambda \Big[\sum_{n=1}^{\tau_\alpha} f(\Phi_n) \Big] \tag{14.14}$$

$$\Big\{ \sum_{n=1}^{\infty} \int \lambda(dx) |a_x * u\,(n) - u(n)| \Big\} \Big\{ \sum_{n=1}^{\infty} {}_{\alpha}P^n(\boldsymbol{\alpha}, f) \Big\}. \tag{14.15}$$

The first of these is again finite since we have assumed λ to be f-regular; and in the second, the right hand term is similarly finite since $\pi(f) < \infty$, whilst the left hand term is independent of f, and since λ is regular (given $f \geq 1$), is bounded by $\mathsf{E}_\lambda[\tau_\alpha] \mathrm{Var}\,(u)$, using (13.73).

Since for some finite M,

$$\mathsf{E}_x[\tau_\alpha] \leq \mathsf{E}_x[\sum_{n=1}^{\tau_\alpha} f(\Phi_n)] \leq M G_\alpha(x, f)$$

this completes the proof of (iii). □

Thus for a chain with an accessible atom, we have very little difficulty moving to f-norm convergence. The simplicity of the results is exemplified in the countable state space case where the f-regularity of all states, guaranteed by Proposition 14.1.2, gives us

Theorem 14.1.4 *Suppose that $\boldsymbol{\Phi}$ is an irreducible positive Harris aperiodic chain on a countable space. Then if $\pi(f) < \infty$, for all $x, y \in \mathsf{X}$*

$$\|P^k(x,\,\cdot\,) - \pi\|_f \to 0 \qquad k \to \infty.$$

and

$$\sum_{n=1}^{\infty} \|P^n(x,\,\cdot\,) - P^n(y,\,\cdot\,)\|_f < \infty.$$

14.2 f-Regularity and drift

It would seem at this stage that all we have to do is move, as we did in Chapter 13, to strongly aperiodic chains; bring the f-properties proved in the previous section above over from the split chain in this case; and then move to general aperiodic chains by using the Nummelin splitting of the m-skeleton.

Somewhat surprisingly, perhaps, this recipe does not work in a trivially easy way. The most difficult step in this approach is that when we go to a split chain it is necessary to consider an m-skeleton, but we do not yet know if the skeletons of an f-regular chain are also f-regular. Such is indeed the case and we will prove this key result in the next section, by exploiting drift criteria.

This may seem to be a much greater effort than we needed for the Aperiodic Ergodic Theorem: but it should be noted that we devoted all of Chapter 11 to the equivalence of regularity and drift conditions in the case of $f \equiv 1$, and the results here actually require rather less effort. In fact, much of the work in this chapter is based on the results already established in Chapter 11, and the duality between drift and regularity established there will serve us well in this more complex case.

14.2.1 The drift characterization of f-regularity

In order to establish f-regularity for a chain on a general state space without atoms, we will use the following criterion, which is a generalization of the condition in (V2). As for regular chains, we will find that there is a duality between appropriate solutions to (V3) and f-regularity.

f-Modulated Drift Towards C

(V3) For a function $f\colon \mathsf{X} \to [1, \infty)$, a set $C \in \mathcal{B}(\mathsf{X})$, a constant $b < \infty$, and an extended-real valued function $V\colon \mathsf{X} \to [0, \infty]$

$$\Delta V(x) \le -f(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{14.16}$$

The condition (14.16) is implied by the slightly stronger pair of bounds

$$f(x) + PV(x) \le \begin{cases} V(x) & x \in C^c \\ b & x \in C \end{cases} \tag{14.17}$$

with V bounded on C, and it is this form that is often verified in practice.

Those states x for which $V(x)$ is finite when V satisfies (V3) will turn out to be those f-regular states from which the distributions of $\boldsymbol{\Phi}$ converge in f-norm. For this reason the following generalization of Lemma 11.3.6 is important: we omit the proof which is similar to that of Lemma 11.3.6 or Proposition 14.1.2.

Lemma 14.2.1 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible. If (14.16) holds for a positive function V which is finite at some $x_0 \in \mathsf{X}$ then the set $S_f := \{x \in \mathsf{X} : V(x) < \infty\}$ is absorbing and full.*

□

The power of (V3) largely comes from the following

Theorem 14.2.2 (Comparison Theorem) *Suppose that the non-negative functions V, f, s satisfy the relationship*

$$PV(x) \le V(x) - f(x) + s(x), \qquad x \in \mathsf{X}$$

Then for each $x \in \mathsf{X}$, $N \in \mathbb{Z}_+$, and any stopping time τ we have

$$\begin{aligned} \sum_{k=0}^{N} \mathsf{E}_x[f(\Phi_k)] &\leq V(x) + \sum_{k=0}^{N} \mathsf{E}_x[s(\Phi_k)] \\ \mathsf{E}_x\Big[\sum_{k=0}^{\tau-1} f(\Phi_k)\Big] &\leq V(x) + \mathsf{E}_x\Big[\sum_{k=0}^{\tau-1} s(\Phi_k)\Big]. \end{aligned}$$

PROOF This follows from Proposition 11.3.2 on letting $f_k = f$, $s_k = s$. □

The first inequality in Theorem 14.2.2 bounds the mean value of $f(\Phi_k)$, but says nothing about the convergence of the mean value. We will see that the second bound is in fact crucial for obtaining f-regularity for the chain, and we turn to this now.

In linking the drift condition (V3) with f-regularity we will consider the extended-real valued function $G_C(x, f)$ defined in (11.21) as

$$G_C(x,f) = \mathsf{E}_x\Big[\sum_{k=0}^{\sigma_C} f(\Phi_k)\Big] \tag{14.18}$$

where C is typically f-regular or petite. The following characterization of f-regularity shows that this function is both a solution to (14.16), and can be bounded using any other solution V to (14.16). Together with Lemma 14.2.1, this result proves the equivalence between (ii) and (iii) in the f-Norm Ergodic Theorem.

Theorem 14.2.3 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible.*

(i) *If (V3) holds for a petite set C then for any $B \in \mathcal{B}^+(\mathsf{X})$ there exists $c(B) < \infty$ such that*

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} f(\Phi_k)\Big] \leq V(x) + c(B).$$

Hence if V is bounded on the set A, then A is f-regular.

(ii) *If there exists one f-regular set $C \in \mathcal{B}^+(\mathsf{X})$, then C is petite and the function $V(x) = G_C(x, f)$ satisfies (V3) and is bounded on A for any f-regular set A.*

PROOF **(i)** Suppose that (V3) holds, with C a ψ_a-petite set. By the Comparison Theorem 14.2.2, Lemma 11.3.10, and the bound

$$\mathbb{1}_C(x) \leq \psi_a(B)^{-1} K_a(x,B)$$

in (11.27) we have for any $B \in \mathcal{B}^+(\mathsf{X})$, $x \in \mathsf{X}$,

$$\begin{aligned} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} f(\Phi_k)\Big] &\leq V(x) + b\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \mathbb{1}_C(\Phi_k)\Big] \\ &\leq V(x) + b\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \psi_a(B)^{-1}K_a(\Phi_k,B)\Big] \\ &= V(x) + b\psi_a(B)^{-1}\sum_{i=0}^{\infty} a_i \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \mathbb{1}_B(\Phi_{k+i})\Big] \\ &\leq V(x) + b\psi_a(B)^{-1}\sum_{i=0}^{\infty} i a_i. \end{aligned}$$

Since we can choose a so that $m_a = \sum_{i=0}^{\infty} i a_i < \infty$ from Proposition 5.5.6, the result follows with $c(B) = b\psi_a(B)^{-1} m_a$. We then have

$$\sup_{x \in A} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} f(\Phi_k)\Big] \le \sup_{x \in A} V(x) + c(B),$$

and so if V is bounded on A, it follows that A is f-regular.

(ii) If an f-regular set $C \in \mathcal{B}^+(\mathsf{X})$ exists, then it is also regular and hence petite from Proposition 11.3.8. The function $G_C(x, f)$ is clearly positive, and bounded on any f-regular set A. Moreover, by Theorem 11.3.5 and f-regularity of C it follows that condition (V3) holds with $V(x) = G_C(x, f)$. □

14.2.2 f-regular sets

Theorem 14.2.3 gives a characterization of f-regularity in terms of a drift condition. The next result gives such a characterization in terms of the return times to petite sets, and generalizes Proposition 11.3.14: f-regular sets in $\mathcal{B}^+(\mathsf{X})$ are precisely those petite sets which are "self-f-regular".

Theorem 14.2.4 *When Φ is a ψ-irreducible chain, the following are equivalent:*

(i) *$C \in \mathcal{B}(\mathsf{X})$ is petite and*

$$\sup_{x \in C} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_C - 1} f(\Phi_k)\Big] < \infty; \tag{14.19}$$

(ii) *C is f-regular and $C \in \mathcal{B}^+(\mathsf{X})$.*

Proof To see that (i) implies (ii), suppose that C is petite and satisfies (14.19). By Theorem 11.3.5 we may find a constant $b < \infty$ such that (V3) holds for $G_C(x, f)$. It follows from Theorem 14.2.3 that C is f-regular.

The set C is Harris recurrent under the conditions of (i), and hence lies in $\mathcal{B}^+(\mathsf{X})$ by Proposition 9.1.1.

Conversely, if C is f-regular then it is also petite from Proposition 11.3.8, and if $C \in \mathcal{B}^+(\mathsf{X})$ then $\sup_{x \in C} \mathsf{E}_x[\sum_{k=0}^{\tau_C - 1} f(\Phi_k)] < \infty$ by the definition of f-regularity. □

As an easy corollary to Theorem 14.2.3 we obtain the following generalization of Proposition 14.1.2.

Theorem 14.2.5 *If there exists an f-regular set $C \in \mathcal{B}^+(\mathsf{X})$, then there exists an increasing sequence $\{S_f(n) : n \in \mathbb{Z}_+\}$ of f-regular sets whose union is full. Hence there is a decomposition*

$$\mathsf{X} = S_f \cup N \tag{14.20}$$

where the set S_f is full and absorbing and Φ restricted to S_f is f-regular.

Proof By f-regularity and positivity of C we have, by Theorem 14.2.3 (ii), that (V3) holds for the function $V(x) = G_C(x, f)$ which is bounded on C, and by Lemma 14.2.1 we have that V is finite π-a.e.

The required sequence of f-regular sets can then be taken as

$$S_f(n) := \{x : V(x) \le n\}, \qquad n \ge 1$$

by Theorem 14.2.3. It is a consequence of Lemma 14.2.1 that $S_f = \cup S_f(n)$ is absorbing. □

We now give a characterization of f-regularity using the Comparison Theorem 14.2.2.

Theorem 14.2.6 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible. Then the chain is f-regular if and only if (V3) holds for an everywhere finite function V, and every sublevel set of V is then f-regular.*

Proof From Theorem 14.2.3 (i) we see that if (V3) holds for a finite-valued V then each sublevel set of V is f-regular. This establishes f-regularity of $\boldsymbol{\Phi}$.

Conversely, if $\boldsymbol{\Phi}$ is f-regular then it follows that an f-regular set $C \in \mathcal{B}^+(\mathsf{X})$ exists. The function $V(x) = G_C(x, f)$ is everywhere finite and satisfies (V3), by Theorem 14.2.3 (ii). □

As a corollary to Theorem 14.2.6 we obtain a final characterization of f-regularity of $\boldsymbol{\Phi}$, this time in terms of petite sets:

Theorem 14.2.7 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible. Then the chain is f-regular if and only if there exists a petite set C such that the expectation*

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_C-1} f(\Phi_k)\Big]$$

is finite for each x, and uniformly bounded for $x \in C$.

Proof If the expectation is finite as described in the theorem, then by Theorem 11.3.5 the function $G_C(x, f)$ is everywhere finite, and satisfies (V3) with the petite set C. Hence from Theorem 14.2.6 we see that the chain is f-regular.

For the converse take C to be any f-regular set in $\mathcal{B}^+(\mathsf{X})$. □

14.2.3 f-Regularity and m-skeletons

One advantage of the form (V3) over (14.17) is that, once f-regularity of $\boldsymbol{\Phi}$ is established, we may easily iterate (14.16) to obtain

$$P^m V(x) \le V(x) - \sum_{i=0}^{m-1} P^i f + \sum_{i=0}^{m-1} P^i \mathbb{1}_C(x) \qquad x \in \mathsf{X}. \tag{14.21}$$

This is essentially of the same form as (14.16), and provides an approach to f-regularity for the m-skeleton which will give us the desired equivalence between f-regularity for $\boldsymbol{\Phi}$ and its skeletons.

To apply Theorem 14.2.3 and (14.21) to obtain an equivalence between f-properties of $\boldsymbol{\Phi}$ and its skeletons we must replace the function $\sum_{i=0}^{m-1} P^i \mathbb{1}_C$ with the indicator function of a petite set. The following result shows that this is possible whenever C is petite and the chain is aperiodic.

Let us write for any positive function g on X,

$$g^{(m)} := \sum_{i=0}^{m-1} P^i g. \tag{14.22}$$

Lemma 14.2.8 *If Φ is aperiodic and if $C \in \mathcal{B}(\mathsf{X})$ is a petite set, then for any $\varepsilon > 0$ and $m \geq 1$ there exists a petite set C_ε such that*

$$\mathbb{1}_C^{(m)} \leq m\mathbb{1}_{C_\varepsilon} + \varepsilon.$$

PROOF Since Φ is aperiodic, it follows from the definition of the period given in (5.40) and the fact that petite sets are small, proven in Proposition 5.5.7, that for a non-trivial measure ν and some $k \in \mathbb{Z}_+$, we have the simultaneous bound

$$P^{km-i}(x,B) \geq \mathbb{1}_C(x)\nu(B), \quad x \in \mathsf{X},\ B \in \mathcal{B}(\mathsf{X}), \qquad 0 \leq i \leq m-1.$$

Hence we also have

$$P^{km}(x,B) \geq P^i\mathbb{1}_C(x)\nu(B), \qquad x \in \mathsf{X},\ B \in \mathcal{B}(\mathsf{X}), \qquad 0 \leq i \leq m-1,$$

which shows that

$$P^{km}(x,\,\cdot\,) \geq \mathbb{1}_C^{(m)}(x)m^{-1}\nu.$$

The set $C_\varepsilon = \{x : \mathbb{1}_C^{(m)}(x) \geq \varepsilon\}$ is therefore ν_k-small for the m-skeleton, where $\nu_k = \varepsilon m^{-1}\nu$, whenever this set is non-empty. Moreover, $C \subset C_\varepsilon$ for all $\varepsilon < 1$.

Since $\mathbb{1}_C^{(m)} \leq m$ everywhere, and since $\mathbb{1}_C^{(m)}(x) < \varepsilon$ for $x \in C_\varepsilon^c$, we have the bound

$$\mathbb{1}_C^{(m)} \leq m\mathbb{1}_{C_\varepsilon} + \varepsilon$$

□

We can now put these pieces together and prove the desired solidarity for Φ and its skeletons.

Theorem 14.2.9 *Suppose that Φ is ψ-irreducible and aperiodic. Then $C \in \mathcal{B}^+(\mathsf{X})$ is f-regular if and only if it is $f^{(m)}$-regular for any one, and then every, m-skeleton chain.*

PROOF If C is $f^{(m)}$-regular for an m-skeleton then, letting τ_B^m denote the hitting time for the skeleton, we have by the Markov property, for any $B \in \mathcal{B}^+(\mathsf{X})$,

$$\begin{aligned} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B^m-1}\sum_{i=0}^{m-1} P^i f(\Phi_{km})\Big] &= \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B^m-1}\sum_{i=0}^{m-1} f(\Phi_{km+i})\Big] \\ &\geq \mathsf{E}_x\Big[\sum_{j=0}^{\tau_B-1} f(\Phi_j)\Big]. \end{aligned}$$

By the assumption of $f^{(m)}$-regularity, the left hand side is bounded over C and hence the set C is f-regular.

Conversely, if $C \in \mathcal{B}^+(\mathsf{X})$ is f-regular then it follows from Theorem 14.2.3 that (V3) holds for a function V which is bounded on C.

By repeatedly applying P to both side of this inequality we obtain as in (14.21)

$$P^mV \leq V - f^{(m)} + b\mathbb{1}_C^{(m)}.$$

By Lemma 14.2.8 we have for a petite set C'

$$\begin{aligned} P^m V &\le V - f^{(m)} + bm\mathbb{1}_{C'} + \tfrac{1}{2} \\ &\le V - \tfrac{1}{2} f^{(m)} + bm\mathbb{1}_{C'}, \end{aligned}$$

and thus (V3) holds for the m-skeleton. Since V is bounded on C, we see from Theorem 14.2.3 that C is $f^{(m)}$-regular for the m-skeleton. □

As a simple but critical corollary we have

Theorem 14.2.10 *Suppose that Φ is ψ-irreducible and aperiodic. Then Φ is f-regular if and only if each m-skeleton is $f^{(m)}$-regular.* □

The importance of this result is that it allows us to shift our attention to skeleton chains, one of which is always strongly aperiodic and hence may be split to form an artificial atom; and this of course allows us to apply the results obtained in Section 14.1 for chains with atoms.

The next result follows this approach to obtain a converse to Proposition 14.1.1, thus extending Proposition 14.1.2 to the non-atomic case.

Theorem 14.2.11 *Suppose that Φ is positive recurrent and $\pi(f) < \infty$. Then there exists a sequence $\{S_f(n)\}$ of f-regular sets whose union is full.*

PROOF We need only look at a split chain corresponding to the m-skeleton chain, which possess an $f^{(m)}$-regular atom by Proposition 14.1.2. It follows from Proposition 14.1.2 that for the split chain the required sequence of $f^{(m)}$-regular sets exist, and then following the proof of Proposition 11.1.3 we see that for the m-skeleton an increasing sequence $\{S_f(n)\}$ of $f^{(m)}$-regular sets exists whose union is full.

From Theorem 14.2.9 we have that each of the sets $\{S_f(n)\}$ is also f-regular for Φ and the theorem is proved. □

14.3 f-Ergodicity for general chains

14.3.1 The aperiodic f-ergodic theorem

We are now, at last, in a position to extend the atom-based f-ergodic results of Section 14.1 to general aperiodic chains.

We first give an f-ergodic theorem for strongly aperiodic chains. This is an easy consequence of the result for chains with atoms.

Proposition 14.3.1 *Suppose that Φ is strongly aperiodic, positive recurrent, and suppose that $f \ge 1$.*

(i) *If $\pi(f) = \infty$ then $P^k(x, f) \to \infty$ as $k \to \infty$ for all $x \in \mathsf{X}$.*

(ii) *If $\pi(f) < \infty$ then almost every state is f-regular and for any f-regular state $x \in \mathsf{X}$*

$$\|P^k(x, \cdot\,) - \pi\|_f \to 0, \qquad k \to \infty.$$

(iii) *If Φ is f-regular then Φ is f-ergodic.*

PROOF (i) By positive recurrence we have for x lying in the maximal Harris set H, and any $m \in \mathbb{Z}_+$,

$$\liminf_{k\to\infty} P^k(x,f) \geq \liminf_{k\to\infty} P^k(x, m \wedge f) = \pi(m \wedge f).$$

Letting $m \to \infty$ we see that $P^k(x,f) \to \infty$ for these x. For arbitrary $x \in \mathsf{X}$ we choose n_0 so large that $P^{n_0}(x,H) > 0$. This is possible by ψ-irreducibility. By Fatou's Lemma we then have the bound

$$\liminf_{k\to\infty} P^k(x,f) = \liminf_{k\to\infty} P^{n_0+k}(x,f) \geq \int_H P^{n_0}(x,dy)\Big\{\liminf_{k\to\infty} P^k(x,f)\Big\} = \infty.$$

Result (ii) is now obvious using the split chain, given the results for a chain possessing an atom, and (iii) follows directly from (ii). □

We again obtain f-ergodic theorems for general aperiodic $\boldsymbol{\Phi}$ by considering the m-skeleton chain. The results obtained in the previous section show that when $\boldsymbol{\Phi}$ has appropriate f-properties then so does each m-skeleton. For aperiodic chains, there always exists some $m \geq 1$ such that the m-skeleton is strongly aperiodic, and hence we may apply Theorem 14.3.1 to the m-skeleton chain to obtain f-ergodicity for this skeleton. This then carries over to the process by considering the m distinct skeleton chains embedded in $\boldsymbol{\Phi}$.

The following lemma allows us to make the desired connections between $\boldsymbol{\Phi}$ and its skeletons.

Lemma 14.3.2 **(i)** *For any $f \geq 1$ we have for $n \in \mathbb{Z}_+$,*

$$\|P^n(x,\,\cdot\,) - \pi\|_f \leq \|P^{km}(x,\cdot) - \pi(\cdot)\|_{f^{(m)}},$$

for k satisfying $n = km + i$ with $0 \leq i \leq m-1$.

(ii) *If for some $m \geq 1$ and some $x \in \mathsf{X}$ we have $\|P^{km}(x,\,\cdot\,) - \pi\|_{f^{(m)}} \to 0$ as $k \to \infty$ then $\|P^k(x,\,\cdot\,) - \pi\|_f \to 0$ as $k \to \infty$.*

(iii) *If the m-skeleton is $f^{(m)}$-ergodic then $\boldsymbol{\Phi}$ itself is f-ergodic.*

PROOF Under the conditions of (i) let $|g| \leq f$ and write any $n \in \mathbb{Z}_+$ as $n = km + i$ with $0 \leq i \leq m-1$. Then

$$\begin{aligned} |P^n(x,g) - \pi(g)| &= |P^{km}(x,P^i g) - \pi(P^i g)| \\ &\leq \|P^{km}(x,\cdot) - \pi(\cdot)\|_{f^{(m)}}. \end{aligned}$$

This proves (i) and the remaining results then follow. □

This lemma and the ergodic theorems obtained for strongly aperiodic chains finally give the result we seek.

Theorem 14.3.3 *Suppose that $\boldsymbol{\Phi}$ is positive recurrent and aperiodic.*

(i) *If $\pi(f) = \infty$ then $P^k(x,f) \to \infty$ for all x.*

(ii) *If $\pi(f) < \infty$ then the set S_f of f-regular sets is full and absorbing, and if $x \in S_f$ then $\|P^k(x,\,\cdot\,) - \pi\|_f \to 0$, as $k \to \infty$.*

(iii) *If $\boldsymbol{\Phi}$ is f-regular then $\boldsymbol{\Phi}$ is f-ergodic. Conversely, if $\boldsymbol{\Phi}$ is f-ergodic then $\boldsymbol{\Phi}$ restricted to a full absorbing set is f-regular.*

PROOF Result (i) follows as in the proof of Proposition 14.3.1 (i).

If $\pi(f) < \infty$ then there exists a sequence of f-regular sets $\{S_f(n)\}$ whose union is full. By aperiodicity, for some m, the m-skeleton is strongly aperiodic and each of the sets $\{S_f(n)\}$ is $f^{(m)}$-regular. From Proposition 14.3.1 we see that the distributions of the m-skeleton converge in $f^{(m)}$-norm for initial $x \in S_f(n)$.

This and Lemma 14.3.2 proves (ii). The first part of (iii) is then a simple consequence; the converse is also immediate from (ii) since f-ergodicity implies $\pi(f) < \infty$. □

Note that if $\boldsymbol{\Phi}$ is f-ergodic then $\boldsymbol{\Phi}$ may not be f-regular: this is already obvious in the case $f = 1$.

14.3.2 Sums of transition probabilities

We now refine the ergodic theorem Theorem 14.3.3 to give conditions under which the sum

$$\sum_{n=1}^{\infty} \|P^n(x, \cdot\,) - \pi\|_f \tag{14.23}$$

is finite.

The first result of this kind requires f-regularity of the initial probability measures λ, μ. For practical implementation, note that if (V3) holds for a petite set C and a function V, and if $\lambda(V) < \infty$, then from Theorem 14.2.3 (i) we see that the measure λ is f-regular.

Theorem 14.3.4 *Suppose $\boldsymbol{\Phi}$ is an aperiodic positive Harris chain. If $\pi(f) < \infty$ then for any f-regular set $C \in \mathcal{B}^+(\mathsf{X})$ there exists $M_f < \infty$ such that for any f-regular initial distributions λ, μ,*

$$\sum_{n=1}^{\infty} \int\int \lambda(dx)\mu(dy)\|P^n(x, \cdot\,) - P^n(y, \cdot\,)\|_f \le M_f(\lambda(V) + \mu(V) + 1) < \infty \tag{14.24}$$

where $V(\,\cdot\,) = G_C(\,\cdot\,, f)$.

PROOF Consider first the strongly aperiodic case, and construct a split chain $\check{\boldsymbol{\Phi}}$ using an f-regular set C. The theorem is valid from Theorem 14.1.3 for the split chain, since the split measures μ^*, λ^* are f-regular for $\check{\boldsymbol{\Phi}}$. The bound on the sum can be taken as

$$\sum_{n=1}^{\infty} \int\int \lambda^*(dx)\mu^*(dy)\|\check{P}^n(x, \cdot\,) - \check{P}^n(y, \cdot\,)\|_f < M_f(\lambda^*(V) + \mu^*(V) + 1)$$

with $V = \check{G}_{C_0 \cup C_1}(\,\cdot\,, f)$, since $C_0 \cup C_1 \in \mathcal{B}^+(\check{\mathsf{X}})$ is f-regular for the split chain.

Since the result is a total variation result it is then obviously valid when restricted to the original chain, as in (13.58). Using the identity

$$\int \lambda^*(dx)\check{G}_{C_0 \cup C_1}(x, f) = \int \lambda(dx)G_C(x, f),$$

and the analogous identity for μ, we see that the required bound holds in the strongly aperiodic case.

In the arbitrary aperiodic case we can apply Lemma 14.3.2 to move to a skeleton chain, as in the proof of Theorem 14.3.3. □

The most interesting special case of this result is given in the following theorem.

Theorem 14.3.5 *Suppose Φ is an aperiodic positive Harris chain and that π is f-regular. Then $\pi(f) < \infty$ and for any f-regular set $C \in \mathcal{B}^+(\mathsf{X})$ there exists $B_f < \infty$ such that for any f-regular initial distribution λ*

$$\sum_{n=1}^{\infty} \|\lambda P^n - \pi\|_f \le B_f(\lambda(V) + 1). \tag{14.25}$$

where $V(\,\cdot\,) = G_C(\,\cdot\,, f)$. □

Our final f-ergodic result, for quite arbitrary positive recurrent chains is given for completeness in

Theorem 14.3.6 **(i)** *If Φ is positive recurrent and if $\pi(f) < \infty$ then there exists a full set S_f, a cycle $\{D_i : 1 \le i \le d\}$ contained in S_f, and probabilities $\{\pi_i : 1 \le i \le d\}$ such that for any $x \in D_r$,*

$$\|P^{nd+r}(x,\,\cdot\,) - \pi_r\|_f \to 0, \qquad n \to \infty. \tag{14.26}$$

(ii) *If Φ is f-regular then for all x,*

$$\|d^{-1}\sum_{r=1}^{d} P^{nd+r}(x,\,\cdot\,) - \pi\|_f \to 0, \qquad n \to \infty. \tag{14.27}$$

□

14.3.3 A criterion for finiteness of $\pi(f)$

From the Comparison Theorem 14.2.2 and the ergodic theorems presented above we also obtain the following criterion for finiteness of moments.

Theorem 14.3.7 *Suppose that Φ is positive recurrent with invariant probability π, and suppose that V, f and s are non-negative, finite-valued functions on X such that*

$$PV(x) \le V(x) - f(x) + s(x)$$

for every $x \in \mathsf{X}$. Then $\pi(f) \le \pi(s)$.

PROOF For π-a.e. $x \in \mathsf{X}$ we have from the Comparison Theorem 14.2.2, Theorem 14.3.6 and (if $\pi(f) = \infty$) the aperiodic version of Theorem 14.3.3, whether or not $\pi(s) < \infty$,

$$\pi(f) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathsf{E}_x[f(\Phi_k)] \le \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathsf{E}_x[s(\Phi_k)] = \pi(s).$$

□

The criterion for $\pi(\mathsf{X}) < \infty$ in Theorem 11.0.1 is a special case of this result. However, it seems easier to prove for quite arbitrary non-negative f, s using these limiting results.

14.4 f-Ergodicity of specific models

14.4.1 Random walk on $\mathbb{R}_+$ and storage models

Consider random walk on a half line given by $\Phi_n = [\Phi_{n-1} + W_n]^+$, and assume that the increment distribution Γ has negative first moment and a finite absolute moment $\sigma^{(k)}$ of order k.

Let us choose the test function $V(x) = x^k$. Then using the binomial expansion the drift Δ_V is given for $x > 0$ by

$$\begin{aligned} \Delta V(x) &= \int_{-x}^{\infty} \Gamma(dy)(x+y)^k - x^k \\ &\leq \Big(\int_{-x}^{\infty} \Gamma(dy) y\Big) k x^{k-1} + c\sigma^{(k)} x^{k-2} + d \end{aligned} \tag{14.28}$$

for some finite c, d. We can rewrite (14.28) in the form of (V3); namely for some $c' > 0$, and large enough x

$$\int P(x,dy) y^k \leq x^k - c' x^{k-1}.$$

From this we may prove the following

Proposition 14.4.1 *If the increment distribution Γ has mean $\beta < 0$ and finite $(k+1)^{st}$ moment, then the associated random walk on a half line is $|x|^k$-regular. Hence the process $\boldsymbol{\Phi}$ admits a stationary measure π with finite moments of order k; and with $f_k(y) = y^k + 1$,*

(i) *for all λ such that $\int \lambda(dx) x^{k+1} < \infty$,*

$$\int \lambda(dx) \| P^n(x,\,\cdot\,) - \pi \|_{f_k} \to 0, \qquad n \to \infty;$$

(ii) *for some $B_f < \infty$, and any initial distribution λ,*

$$\sum_{n=0}^{\infty} \int \lambda(dx) \| P^n(x,\,\cdot\,) - \pi \|_{f_{k-1}} \leq B_f \Big(1 + \int x^k\, \lambda(dx)\Big)$$

Proof The calculations preceding the proposition show that for some $c_0 > 0$, $d_0 < \infty$, and a compact set $C \subset \mathbb{R}_+$,

$$PV_{i+1}(x) \leq V_{i+1}(x) - c_0 f_i(x) + d_0 \mathbb{1}_C(x) \qquad 0 \leq i \leq k, \tag{14.29}$$

where $V_j(x) = x^j$, $f_j(x) = x^j + 1$. Result (i) is then an immediate consequence of the f-Norm Ergodic Theorem.

To prove (ii) apply (14.29) with $i = k$ and Theorem 14.3.7 to conclude that $\pi(V_k) < \infty$. Applying (14.29) again with $i = k-1$ we see that π is f_{k-1}-regular and then (ii) follows from the f-Norm Ergodic Theorem. □

It is well known that the invariant measure for a random walk on the half line has moments of order one degree lower than those of the increment distribution, but this is a particularly simple proof of this result.

For the Moran dam model or the queueing models developed in Chapter 2, this result translates directly into a condition on the input distribution. Provided the mean input is less than the mean output between input times, then there is a finite invariant measure: and this has a finite k^{th} moment if the input distribution has finite $(k+1)^{st}$ moment.

14.4.2 Bilinear Models

The random walk model in the previous section can be generalized in a variety of ways, as we have seen many times in the applications above.

For illustrative purposes we next consider the scalar bilinear model

$$X_{k+1} = \theta X_k + bW_{k+1}X_k + W_{k+1} \tag{14.30}$$

for which we proved boundedness in probability in Section 12.5.2. For simplicity, we take $\mathsf{E}[W] = 0$.

To obtain a solution to (V3), assume that $\mathbf{W}$ has finite variance. Then for the test function $V(x) = x^2$, we observe that by independence

$$\mathsf{E}[(X_{k+1})^2 \mid X_k = x] \le \Big[\theta^2 + b^2\mathsf{E}[W_{k+1}^2]\Big]x^2 + (2bx+1)\mathsf{E}[W_{k+1}^2]. \tag{14.31}$$

Since this V is a norm-like function on $\mathbb{R}$, it follows that (V3) holds with the choice of

$$f(x) = 1 + \delta V(x)$$

for some $\delta > 0$ provided

$$\theta^2 + b^2\mathsf{E}[W_k^2] < 1. \tag{14.32}$$

Under this condition it follows just as in the LSS(F) model that provided the noise process forces this model to be a T-chain (for example, if the conditions of Proposition 7.1.3 hold) then (14.32) is a condition not just for positive Harris recurrence, but for the existence of a second order stationary model with finite variance: this is precisely the interpretation of $\pi(f) < \infty$ in this case.

A more general version of this result is

Proposition 14.4.2 *Suppose that (SBL1) and (SBL2) hold and*

$$\mathsf{E}[W_n^k] < \infty. \tag{14.33}$$

Then the bilinear model is positive Harris, the invariant measure π also has finite k^{th} moments (that is, satisfies $\int x^k\pi(dx) < \infty$), and

$$\|P^n(x,\,\cdot\,) - \pi\|_{x^k} \to 0, \qquad n \to \infty.$$

□

In the next chapter we will show that there is in fact a geometric rate of convergence in this result. This will show that, in essence, the same drift condition gives us finiteness of moments in the stationary case, convergence of time-dependent moments and some conclusion about the rate at which the moments become stationary.

14.5 A Key Renewal Theorem

One of the most interesting applications of the ergodic theorems in these last two chapters is a probabilistic proof of the Key Renewal Theorem.

As in Section 3.5.3, let $Z_n := \sum_{i=0}^n Y_i$, where $\{Y_1, Y_2, \ldots\}$ is a sequence of independent and identical random variables with distribution Γ on $\mathbb{R}_+$, and Y_0 is a further independent random variable with distribution Γ_0 also on $\mathbb{R}_+$; and let $U(\,\cdot\,) = \sum_{n=0}^{\infty} \Gamma^{n*}(\,\cdot\,)$ be the associated renewal measure.

Renewal theorems concern the limiting behavior of U; specifically, they concern conditions under which

$$\Gamma_0 * U * f\,(t) \to \beta^{-1} \int_0^{\infty} f(s)\; ds \tag{14.34}$$

as $t \to \infty$, where $\beta = \int_0^\infty s\Gamma(ds)$ and f and Γ_0 are an appropriate function and measure respectively.

With minimal assumptions about Γ we have *Blackwell's Renewal Theorem.*

Theorem 14.5.1 *Provided Γ has a finite mean β and is not concentrated on a lattice $nh, n \in \mathbb{Z}_+, h > 0$, then for any interval $[a,b]$ and any initial distribution Γ_0*

$$\Gamma_0 * U[a+t, b+t] \to \beta^{-1}(b-a), \qquad t \to \infty. \tag{14.35}$$

PROOF This result is taken from Feller ([77], p. 360) and its proof is not one we pursue here. We do note that it is a special case of the general Key Renewal Theorem, which states that under these conditions on Γ, (14.34) holds for all bounded non-negative functions f which are *directly Riemann integrable*, for which again see Feller ([77], p. 361); for then (14.35) is the special case with $f(s) = \mathbb{1}_{[a,b]}(s)$. □

This result shows us the pattern for renewal theorems: in the limit, the measure U approximates normalized Lebesgue measure.

We now show that one can trade off properties of Γ against properties of f (and to some extent properties of Γ_0) in asserting (14.34). We shall give a proof, based on the ergodic properties we have been considering for Markov chains, of the following Uniform Key Renewal Theorem.

Theorem 14.5.2 *Suppose that Γ has a finite mean β and is spread out (as defined in (RW2)).*

(a) *For any initial distribution Γ_0 we have the uniform convergence*

$$\lim_{t\to\infty} \sup_{|g|\le f} |\Gamma_0 * U * g(t) - \beta^{-1} \int_0^\infty g(s) ds| = 0 \tag{14.36}$$

provided the function $f \ge 0$ satisfies

$$f \quad \textit{is bounded;} \tag{14.37}$$

$$f \quad \textit{is Lebesgue integrable;} \tag{14.38}$$

$$f(t) \to 0, \qquad t \to \infty. \tag{14.39}$$

(b) *In particular, for any bounded interval $[a,b]$ and Borel sets B*

$$\lim_{t\to\infty} \sup_{B\subseteq[a,b]} |\Gamma_0 * U(t+B) - \beta^{-1}\mu^{\rm Leb}(B)| = 0. \tag{14.40}$$

(c) *For any initial distribution Γ_0 which is absolutely continuous, the convergence (14.36) holds for f satisfying only (14.37) and (14.38).*

PROOF The proof of this set of results occupies the remainder of this section, and contains a number of results of independent interest. □

Before embarking on this proof, we note explicitly that we have accomplished a number of tradeoffs in this result, compared with the Blackwell Renewal Theorem. By considering spread-out distributions, we have exchanged the direct Riemann integrability condition for the simpler and often more verifiable smoothness conditions (14.37)-(14.39). This is exemplified by the fact that (14.40) allows us to consider the renewal measure of any bounded Borel set, whereas the general Γ version restricts us to intervals as in (14.35). The extra benefits of smoothness of Γ_0 in removing (14.39) as a condition are also in this vein.

Moreover, by moving to the class of spread-out distributions, we have introduced a uniformity into the Key Renewal Theorem which is analogous in many ways to the total variation norm result in Markov chain limit theory. This analogy is not coincidental: as we now show, these results are all consequences of precisely that total variation convergence for the forward recurrence chain associated with this renewal process.

Recall from Section 3.5.3 the forward recurrence time process

$$V^+(t) := \inf(Z_n - t : Z_n \geq t), \qquad t \geq 0.$$

We will consider the forward recurrence time δ-skeleton $\mathbf{V}_\delta^+ = V^+(n\delta)$, $n \in \mathbb{Z}_+$ for that process, and denote its n-step transition law by $P^{n\delta}(x, \cdot)$. We showed that for sufficiently small δ, when Γ is spread out, then (Proposition 5.3.3) the set $[0, \delta]$ is a small set for $\mathbf{V}_\delta^+$, and (Proposition 5.4.7) $\mathbf{V}_\delta^+$ is also aperiodic.

It is trivial for this chain to see that (V2) holds with $V(x) = x$, so that the chain is regular from Theorem 11.3.15, and if Γ_0 has a finite mean, then Γ_0 is regular from Theorem 11.3.12.

This immediately enables us to assert from Theorem 13.4.4 that, if Γ_1, Γ_2 are two initial measures both with finite mean, and if Γ itself is spread out with finite mean,

$$\sum_{n=0}^{\infty} \| \Gamma_1 P^{n\delta}(\,\cdot\,) - \Gamma_2 P^{n\delta}(\,\cdot\,) \| < \infty. \tag{14.41}$$

The crucial corollary to this example of Theorem 13.4.4, which leads to the Uniform Key Renewal Theorem is

Proposition 14.5.3 *If Γ is spread out with finite mean, and if Γ_1, Γ_2 are two initial measures both with finite mean, then*

$$\| \Gamma_1 * U - \Gamma_2 * U \| := \int_0^\infty | \Gamma_1 * U(dt) - \Gamma_2 * U(dt) | < \infty. \tag{14.42}$$

PROOF By interpreting the measure $\Gamma_0 P^s$ as an initial distribution, observe that for $A \subseteq [t, \infty)$, and fixed $s \in [0, t)$, we have from the Markov property at s the identity

$$\Gamma_0 * U(A) = \Gamma_0 P^s * U(A - s). \tag{14.43}$$

Using this we then have

$$
\begin{aligned}
\int_0^\infty & |\Gamma_1 * U(dt) - \Gamma_2 * U(dt)| \\
&= \sum_{n=0}^\infty \int_{[n\delta,(n+1)\delta)} |\Gamma_1 * U(dt) - \Gamma_2 * U(dt)| \\
&= \sum_{n=0}^\infty \int_{[0,\delta)} |(\Gamma_1 P^{n\delta} - \Gamma_2 P^{n\delta}) * U(dt)| \\
&\leq \sum_{n=0}^\infty \int_{[0,\delta)} \int_{[0,t]} |(\Gamma_1 P^{n\delta} - \Gamma_2 P^{n\delta})(du)| U(dt-u) \\
&\leq \sum_{n=0}^\infty \int_{[0,\delta)} |(\Gamma_1 P^{n\delta} - \Gamma_2 P^{n\delta})(du)| U[0,\delta) \\
&\leq U[0,\delta) \sum_{n=0}^\infty \|\Gamma_1 P^{n\delta} - \Gamma_2 P^{n\delta}\|
\end{aligned}
\tag{14.44}
$$

which is finite from (14.41). □

From this we can prove a precursor to Theorem 14.5.2.

Proposition 14.5.4 *If Γ is spread out with finite mean, and if Γ_1, Γ_2 are two initial measures both with finite mean, then*

$$\sup_{|g|\leq f} |\Gamma_1 * U * g(t) - \Gamma_2 * U * g(t)| \to 0, \qquad t \to \infty \tag{14.45}$$

for any f satisfying (14.37)-(14.39).

Proof Suppose that ε is arbitrarily small but fixed. Using Proposition 14.5.3 we can fix T such that

$$\int_T^\infty |(\Gamma_1 * U - \Gamma_2 * U)(du)| \leq \varepsilon. \tag{14.46}$$

If f satisfies (14.39), then for all sufficiently large t,

$$f(t-u) \leq \varepsilon, \qquad u \in [0,T];$$

for such a t, writing $d = \sup f(x) < \infty$ from (14.37), it follows that for any g with $|g| \leq f$,

$$
\begin{aligned}
|\Gamma_1 * U * g(t) - \Gamma_2 * U * g(t)| &\leq \int_0^T |(\Gamma_1 * U - \Gamma_2 * U(du)| f(t-u) \\
&\quad + \int_T^t |(\Gamma_1 * U - \Gamma_2 * U)(du)| f(t-u) \\
&\leq \varepsilon \|\Gamma_1 * U - \Gamma_2 * U\| + \varepsilon d \\
&:= \varepsilon'
\end{aligned}
\tag{14.47}
$$

which is arbitrarily small, from (14.44), thus proving the result. □

This would prove Theorem 14.5.2 (a) if the equilibrium measure

$$\Gamma_e[0,t] = \beta^{-1} \int_0^t \Gamma(u,\infty) du$$

defined in (10.37) were itself regular, since we have that $\Gamma_e * U(\,\cdot\,) = \beta^{-1}\mu^{\text{Leb}}(\,\cdot\,)$, which gives the right hand side of (14.36). But as can be verified by direct calculation, Γ_e is regular if and only if Γ has a finite second moment, exactly as is the case in Theorem 13.4.5 for general chains with atoms.

However, we can reach the following result, of which Theorem 14.5.2 (a) is a corollary, using a truncation argument.

Proposition 14.5.5 *If Γ is spread out with finite mean, and if Γ_1, Γ_2 are any two initial measures, then*

$$\sup_{|g|\leq f} |\Gamma_1 * U * g(t) - \Gamma_2 * U * g(t)| \to 0, \qquad t\to\infty$$

for any f satisfying (14.37)-(14.39).

PROOF For fixed v, let $\Gamma^v(A){:=}\Gamma(A)/\Gamma[0,v]$ for all $A\subseteq[0,v]$ denote the truncation of $\Gamma(A)$ to $[0,v]$.

For any g with $|g|\leq f$,

$$|\Gamma_1 * U * g(t) - \Gamma_1^v * U * g(t)| \leq \|\Gamma_1 - \Gamma_1^v\| \sup_x U * f(x) \tag{14.48}$$

which can be made smaller than ε by choosing v large enough, provided $\sup_x U * f(x)<\infty$. But if $t>T$, from (14.47), with $\Gamma_1=\delta_0$, $\Gamma_2=\Gamma_e^v$ and $g=f$,

$$\begin{aligned} U * f(t) &= \delta_0 * U * f(t) \\ &\leq \Gamma_e^v * U * f(t) + \varepsilon' \\ &\leq \Big(\Gamma_e[0,v]\Big)^{-1}\Gamma_e * U * f(t) + \varepsilon' \\ &\leq \Big(\Gamma_e[0,v]\Big)^{-1}\beta^{-1}\textstyle\int_0^\infty f(u)du + \varepsilon' \end{aligned} \tag{14.49}$$

which is indeed finite, by (14.38).

The result then follows from Proposition 14.5.4 and (14.48) by a standard triangle inequality argument. □

Theorem 14.5.2 (b) is a simple consequence of Theorem 14.5.2 (a), but to prove Theorem 14.5.2 (c), we need to refine the arguments above a little.

Suppose that (14.39) does not hold, and write

$$A_\varepsilon(t) := \{u\in[0,T] : f(t-u)\geq\varepsilon\}$$

where ε and T are as in (14.46). We then have

$$\begin{aligned} &\int_0^T |(\Gamma_1 * U - \Gamma_2 * U)(du)| f(t-u) \\ &\leq \textstyle\int_0^T |(\Gamma_1 * U - \Gamma_2 * U(du)| f(t-u)\mathbb{1}_{[A_\varepsilon(t)]^c}(u) \\ &\qquad + \textstyle\int_0^T (\Gamma_1 * U + \Gamma_2 * U)(du) f(t-u)\mathbb{1}_{A_\varepsilon(t)}(u) \\ &\leq \varepsilon\|\Gamma_1 * U - \Gamma_2 * U\| + d(\Gamma_1+\Gamma_2) * U(A_\varepsilon(t)). \end{aligned} \tag{14.50}$$

If we now assume that the measure $\Gamma_1+\Gamma_2$ to be absolutely continuous with respect to μ^{Leb}, then so is $(\Gamma_1+\Gamma_2) * U$ ([77], p. 146).

Now since f is integrable, as $t\to\infty$ for fixed T,ε we must have $\mu^{\text{Leb}}(A_\varepsilon(t))\to 0$. But since T is fixed, we have that both $\mu^{\text{Leb}}[0,T]<\infty$ and $(\Gamma_1+\Gamma_2) * U[0,T]<\infty$, and it is a standard result of measure theory ([94], p 125) that

$$(\Gamma_1 + \Gamma_2) * U(A_\varepsilon(t)) \to 0, \qquad t \to \infty.$$

We can thus make the last term in (14.50) arbitrarily small for large t, even without assuming (14.39); now reconsidering (14.47), we see that Proposition 14.5.4 holds without (14.39), provided we assume the existence of densities for Γ_1 and Γ_2, and then Theorem 14.5.2 (c) follows by the truncation argument of Proposition 14.5.5.

14.6 Commentary

These results are largely recent. Although the question of convergence of $\mathsf{E}_x[f(\Phi_k)]$ for general f occurs in, for example, Markov reward models [20], most of the literature on Harris chains has concentrated on convergence only for $f \leq 1$ as in the previous chapter. The results developed here are a more complete form of those in Meyn and Tweedie [178], but there the general aperiodic case was not developed: only the strongly aperiodic case is considered in detail. A more embryonic form of the convergence in f-norm, indicating that if $\pi(f) < \infty$ then $\mathsf{E}_x[f(\Phi_k)] \to \pi(f)$, appeared as Theorem 2 of Tweedie [278].

Nummelin [202] considers f-regularity, but does not go on to apply the resulting concepts to f-ergodicity, although in fact there are connections between the two which are implicit through the Regenerative Decomposition in Nummelin and Tweedie [206].

That Theorem 14.1.1 admits a converse, so that when $\pi(f) < \infty$ there exists a sequence of f-regular sets $\{S_f(n)\}$ whose union is full, is surprisingly deep. For general state space chains, the question of the existence of f-regular sets requires the splitting technique as did the existence of regular sets in Chapter 11. The key to their use in analyzing chains which are not strongly aperiodic lies in the duality with the drift condition (V3), and this is given here for the first time.

The fact that (V3) gives a criterion for finiteness of $\pi(f)$ was observed in Tweedie [278]. Its use for asserting the second order stationarity of bilinear and other time series models was developed in Feigin and Tweedie [74], and for analyzing random walk in [279]. Related results on the existence of moments are also in Kalashnikov [116].

The application to the generalized Key Renewal Theorem is particularly satisfying. By applying the ergodic theorems above to the forward recurrence time chain $\mathbf{V}_\delta^+$, we have "leveraged" from the discrete time renewal theory results of Section 13.2 to the continuous time ones through the general Markov chain results. This Markovian approach was developed in Arjas et al [9], and the uniformity in Theorem 14.5.2, which is a natural consequence of this approach, seems to be new there. The simpler form without the uniformity, showing that one can exchange spread-outness of Γ for the weaker conditions on f dates back to the original renewal theorems of Smith [247, 248, 249], whilst Breiman [30] gives a form of Theorem 14.5.2 (b). An elegant and different approach is also possible through Stone's Decomposition of U [258], which shows that when Γ is spread-out,

$$U = U_f + U_c$$

where U_f is a finite measure, and U_c has a density p with respect to μ^{Leb} satisfying $p(t) \to \beta^{-1}$ as $t \to \infty$.

The convergence, or rather summability, of the quantities

$$\|P^n(x,\,\cdot\,) - \pi\|_f$$

leads naturally to a study of rates of convergence, and this is carried out in Nummelin and Tuominen [205]. Building on this, Tweedie [279] uses similar approaches to those in this chapter to derive drift criteria for more subtle rate of convergence results: the interested reader should note the result of Theorem 3 (iii) of [279]. There it is shown (essentially by using the Comparison Theorem) that if (V3) holds for a function f such that

$$f(x) \geq \mathsf{E}_x[r(\tau_C)], \qquad x \in C^c$$

where $r(n)$ is some function on $\mathbb{Z}_+$, then

$$V(x) \geq \mathsf{E}_x[r^0(\tau_C)], \qquad x \in C^c$$

where $r^0(n) = \sum_1^n r(j)$. If C is petite then this is (see [205] or Theorem 4 (iii) of [279]) enough to ensure that

$$r(n)\|P^n(x,\,\cdot\,) - \pi\| \to 0, \qquad n \to \infty$$

so that (V3) gives convergence at rate $r(n)^{-1}$ in the ergodic theorem.

Applications of these ideas to the Key Renewal Theorem are also contained in [205].

The special case of $r(n) = r^n$ is explored thoroughly in the next two chapters. The rate results above are valuable also in the case of $r(n) = n^k$ since then $r^0(n)$ is asymptotically n^{k+1}. This allows an inductive approach to the level of convergence rate achieved; but this more general topic is not pursued in this book. The interested reader will find the most recent versions, building on those of Nummelin and Tuominen [205], in [271].