18

Positivity

Turning from the sample path and classical limit theorems for normalized sums of the previous chapter, we now return to considering limits of the transition probabilities $P^n(x, A)$.

Our first goal in this chapter is to derive limit theorems for chains which are not positive Harris recurrent. Although some results in this spirit have been derived as ratio limit theorems such as Theorem 17.2.1 and Theorem 17.3.2, we have not to this point considered in any detail the difference between limiting behavior of positive and null recurrent chains.

The last five chapters have amply illustrated the power of ψ-irreducibility in the positive case: that is, in conjunction with the existence of an invariant probability measure. However, even in the non-positive case, powerful and elegant results can be achieved. For Harris recurrent chains we prove a generalized version of the Aperiodic Ergodicity Theorem of Chapter 13, which covers the null recurrent case and actually subsumes the ergodic case also, since it applies to any Harris recurrent chain. We will show

Theorem 18.0.1 *Suppose Φ is an aperiodic Harris recurrent chain. Then for any initial probability distributions λ, μ,*

$$\int\int \lambda(dx)\mu(dy)\|P^n(x,\,\cdot\,) - P^n(y,\,\cdot\,)\| \to 0, \qquad n \to \infty. \tag{18.1}$$

If Φ is a null recurrent chain with invariant measure π, then for any constant $\varepsilon > 0$, and any initial distribution λ

$$\lim_{n\to\infty} \sup_{A\in\mathcal{B}(\mathsf{X})} \int \lambda(dx)P^n(x,A)/[\pi(A)+\varepsilon] = 0. \tag{18.2}$$

PROOF The first result is shown in Theorem 18.1.2 after developing some extended coupling arguments and then applying the splitting technique. The consequence (18.2) is proved in Theorem 18.1.3. □

Our second goal in this chapter is to use these limit results to complete the characterizations of positivity through a positive/null dichotomy of the local behavior of P^n on suitable sets: not surprisingly, the sets of relevance are petite or compact sets in the general or topological settings respectively.

In the classical countable state space analysis, as in say Chung [49] or Feller [76] or Çinlar [40], it is standard to first approach positivity as an asymptotic "P^n-property" of individual states. It is not hard to show that when Φ is irreducible, either

$\limsup_{n\to\infty} P^n(x,y) > 0$ for all $x, y \in \mathsf{X}$ or $\lim_{n\to\infty} P^n(x,y) = 0$ for all $x, y \in \mathsf{X}$. These classifications then provide different but ultimately equivent characterizations of positive and null chains in the sense we have defined them, which is through the finiteness or otherwise of $\pi(\mathsf{X})$. In Theorem 18.2.2 we show that for ψ-irreducible chains the positive/null dichotomy as defined in, say, Theorem 13.0.1 is equivalent to similar dichotomous behavior of

$$\limsup_{n\to\infty} P^n(x, C) \tag{18.3}$$

for petite sets, exactly as it is in the countable case.

Hence for irreducible T-chains, positivity of the chain is characterized by positivity of (18.3) for compact sets C. For T-chains we also show in this chapter that positivity is characterized by the behavior of (18.3) for the open neighborhoods of x, and that similar characterizations exist for e-chains. Thus there are, for these two classes of topologically well-behaved chains, descriptions in topological terms of the various concepts embodied in the concept of positivity.

These results are summarized in the following theorem:

Theorem 18.0.2 *Suppose that Φ is a chain on a topological space for which a reachable state $x^\star \in \mathsf{X}$ exists.*

(i) *If the chain is a T-chain then the following are equivalent:*

- **(a)** *Φ is positive Harris;*
- **(b)** *Φ is bounded in probability;*
- **(c)** *Φ is non-evanescent and $x^\star$ is "positive";*

If any of these equivalent conditions hold and if the chain is aperiodic, then for each initial state $x \in \mathsf{X}$,

$$\|P^k(x,\,\cdot\,) - \pi\| \to 0 \qquad \text{as } k \to \infty. \tag{18.4}$$

(ii) *If the chain is an e-chain then the following are equivalent:*

- **(a)** *There exists a unique invariant probability π and for every initial condition $x \in \mathsf{X}$ and each bounded continuous function $f \in \mathcal{C}(\mathsf{X})$,*

$$\begin{aligned} \lim_{k\to\infty} \overline{P}_k(x,f) &= \pi(f) \\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(\Phi_i) &= \pi(f) \qquad \text{in probability;} \end{aligned}$$

- **(b)** *Φ is bounded in probability on average;*
- **(c)** *Φ is non-evanescent and $x^\star$ is "positive";*

If any of these equivalent conditions hold and if the reachable state is "aperiodic" then for each initial state $x \in \mathsf{X}$,

$$P^k(x,\,\cdot\,) \xrightarrow{\text{w}} \pi \qquad \text{as } k \to \infty. \tag{18.5}$$

PROOF **(i)** The equivalence of Harris positivity and boundedness in probability for T-chains is given in Theorem 18.3.2, and the equivalence of (a) and (c) follows from Proposition 18.3.3.

(ii) The equivalences of (a)-(c) follow from Proposition 18.4.2, and the limit result (18.5) is given in Theorem 18.4.4. □

Thus we have global convergence properties following from local properties, whether the local properties are with respect to petite sets as in Theorem 18.0.1 or neighborhoods of points as in Theorem 18.0.2.

Finally, we revisit the LLN for e-chains in the light of these characterizations and show that a slight strengthening of the hypotheses of Theorem 18.0.2 are precisely those needed for such chains to obey such a law.

18.1 Null recurrent chains

Our initial step in examining positivity is to develop, somewhat paradoxically, a limit result whose main novelty is for null recurrent chains. Orey's Theorem 18.1.2 actually subsumes some aspects of the ergodic theorem in the positive case, but for us its virtue lies in ensuring that limits can be also be defined for null chains.

The method of proof is again via a coupling argument and the Regenerative Decomposition.

The coupling in Section 13.2 was made somewhat easier because of the existence of a finite invariant measure in product form to give positivity of the forward recurrence time chain. If the mean time between renewals is not finite, then such a coupling of independent copies of the renewal process may not actually occur with probability one. To see this, consider the recurrence and transience classification of simple symmetric random walks in two and four dimensions (see Spitzer [255], Section 8). The former is known to be recurrent, so the return times to zero form a proper renewal sequence. Now consider two independent copies of this random walk: this is a four-dimensional random walk which is equally well known to be transient, so the return time to zero is infinite with positive probability.

Since this is the coupling time of the two independent renewal processes, we cannot couple them as we did in the positive recurrent case. It is therefore perhaps surprising that we can achieve our aims by the following rather different and less obvious coupling method.

18.1.1 Coupling renewal processes for null chains

As in Section 13.2 we again define two sets of random variables $\{S_0, S_1, S_2, \dots\}$ and $\{S'_0, S'_1, S'_2, \dots\}$, where $\{S_1, S_2, \dots\}$ are independent and identically distributed with distribution $\{p(j)\}$, and the distributions of the independent variables S_0, S'_0 are a, b.

This time, however we define the second sequence $\{S'_1, S'_2, \dots\}$ in a dependent way. Let M be a (typically large, and yet to be chosen) integer. For each j define S'_j as being either exactly S_j if $S_j > M$, or, if $S_j \le M$, define S'_j as being an independent variable with the same conditional distribution as S_j, namely

$$\mathsf{P}(S'_j = k \mid S_j \le M) = p(k)/(1 - \overline{p}(M)), \qquad k \le M,$$

where $\overline{p}(M) = \sum_{j>M} p(j)$.

This construction ensures that for $j \geq 1$ the increments S_j and S'_j are identical in distribution even though they are not independent. By construction, also, the quantities

$$W_j = S_j - S'_j$$

have the properties that they are identically distributed, they are bounded above by M and below by $-M$, and they are symmetric around zero and in particular have zero mean.

Let $\Phi^*_n = \sum_{j=0}^{n} W_j$ denote the random walk generated by this sequence of variables, and let T^*_{ab} denote the first time that the random walk $\boldsymbol{\Phi}^*$ returns to zero, when the initial step $W_0 = S_0 - S'_0$ has the distribution induced by choosing a, b as the distributions of S_0, S'_0 respectively.

As in Section 13.2 the coupling time of the two renewal processes is defined as

$$T_{ab} = \min\{j : Z_a(j) = Z_b(j) = 1\}$$

where Z_a, Z_b are the indicator sequences of each renewal process, and since

$$\Phi^*_n = \sum_{j=0}^{n} S_j - \sum_{j=0}^{n} S'_j$$

we have immediately that

$$T_{ab} = T^*_{ab}.$$

But we have shown in Proposition 8.4.4 that such a random walk, with its bounded increments, is recurrent on $\mathbb{Z}$, provided of course that it is ψ-irreducible; and if the random walk is recurrent, $T^*_{ab} < \infty$ with probability one from all initial distributions and we have a successful coupling of the two sequences.

Oddly enough, it is now the irreducibility that causes the problems. Obviously a random walk need not be irreducible if the increment distribution Γ is concentrated on sublattices of $\mathbb{Z}$, and as yet we have no guarantee that $\boldsymbol{\Phi}^*$ does not have increments concentrated on such a sublattice: it is clear that it may actually do so without further assumptions.

We now proceed with the proof of the result we require, which is the same conclusion as in Theorem 13.2.2 without the assumption that $m_p < \infty$; and the issues just raised are addressed in that proof.

Theorem 18.1.1 *Suppose that a, b, p are proper distributions on $\mathbb{Z}_+$, and that u is the renewal function corresponding to p. Then provided p is aperiodic*

$$|a * u - b * u|(n) \to 0, \quad n \to \infty, \tag{18.6}$$

and

$$|a * u - b * u| * \overline{p}(n) \to 0, \quad n \to \infty. \tag{18.7}$$

PROOF We will first assume a stronger form of aperiodicity, namely

$$\text{g.c.d.}\{n - m : m < n,\ p(m) > 0,\ p(n) > 0\} = 1.$$

With this assumption we can choose M sufficiently large that

$$\text{g.c.d.}\{n - m : m < n \leq M,\ p(m) > 0,\ p(n) > 0\} = 1. \tag{18.8}$$

Let us use this M in the construction of the random walk $\boldsymbol{\Phi}^*$ above. It is straightforward to check that now $\boldsymbol{\Phi}^*$ really is irreducible, and so

$$\mathsf{P}(T_{ab} < \infty) = 1$$

for any a, b. In particular, then, (18.6) is true for a, b.

We now move on to prove (18.7), and to do this we will now use the backward recurrence chain rather than the forward recurrence chain.

Let V_a^-, V_b^- be the backward recurrence chains defined for the renewal indicators Z_a^-, Z_b^-: note that the subscripts a, b denote conditional random variables with the initial distributions indicated. It is obvious that the chains V_a^-, V_b^- couple at the same time T_{ab} that Z_a^-, Z_b^- couple.

Now let A be an arbitrary set in $\mathbb{Z}_+$. Since the distributions of V_a^- and V_b^- are identical after the time T_{ab} we have for any $n \geq 1$ by decomposing over the values of T_{ab} and using the Markov or renewal property

$$\mathsf{P}(V_a^-(n) \in A) = \sum_{m=1}^{n} \mathsf{P}(T_{ab} = m)\mathsf{P}(V_a^-(n-m) \in A) + \mathsf{P}(V_a^-(n) \in A, T_{ab} > n)$$

$$\mathsf{P}(V_b^-(n) \in A) = \sum_{m=1}^{n} \mathsf{P}(T_{ab} = m)\mathsf{P}(V_b^-(n-m) \in A) + \mathsf{P}(V_b^-(n) \in A, T_{ab} > n).$$

Using this and the inequality $|x - y| \leq \max(x, y)$, $x, y \geq 0$, we get

$$\sup_{A \subseteq \mathbb{Z}_+} |\mathsf{P}(V_a^-(n) \in A) - \mathsf{P}(V_b^-(n) \in A)| \leq \mathsf{P}(T_{ab} > n). \tag{18.9}$$

We already know that the right hand side of (18.9) tends to zero. But the left hand side can be written as

$$\begin{aligned} &\sup_{A \subseteq \mathbb{Z}_+} |\mathsf{P}(V_a^-(n) \in A) - \mathsf{P}(V_b^-(n) \in A)| \\ &= \tfrac{1}{2} \sum_{m=0}^{\infty} |\mathsf{P}(V_a^-(n) = m) - \mathsf{P}(V_b^-(n) = m)| \\ &= \tfrac{1}{2} \sum_{m=0}^{n} |a * u\,(n-m)\overline{p}(m) - b * u\,(n-m)\overline{p}(m)| \\ &= \tfrac{1}{2} |a * u - b * u| * \overline{p}(n) \end{aligned} \tag{18.10}$$

and so the result (18.7) holds.

It remains to remove the extraneous aperiodicity assumption (18.8).

To do this we use a rather nice trick. Let us modify the distribution $p(j)$ to form another distribution $p^0(j)$ on $\{0, 1, \ldots\}$ defined by setting

$$p^0(0) = p > 0;$$

$$p^0(j) = (1-p)p(j), \qquad j \geq 1.$$

Let us now carry out all of the above analysis using p^0, noting that even though this is not a standard renewal sequence since $p^0(0) > 0$, all of the operations used above remain valid.

Provided of course that $p(j)$ is aperiodic in the usual way, we certainly have that (18.8) holds for p^0 and we can conclude that as $n \to \infty$,

$$|a * u^0 - b * u^0|(n) \to 0, \tag{18.11}$$

$$|a * u^0 - b * u^0| * \overline{p}^0(n) \to 0. \tag{18.12}$$

Finally, by construction of p^0 we have the two identities

$$\overline{p}^0(n) = (1-p)\overline{p}(n), \qquad u^0(n) = (1-p)^{-1}u(n)$$

and consequently, from (18.11) and (18.12) we have exactly (18.6) and (18.7) as required. □

Note that in the null recurrent case, since we do not have $\sum \overline{p}(n) < \infty$, we cannot prove this result from Lemma D.7.1 even though it is a identical conclusion to that reached there in the positive recurrent case.

18.1.2 Orey's convergence theorem

In the positive recurrent case, the asymptotic properties of the chain are interesting largely because of the proper distribution π occurring as the limit of the sequence P^n.

In the null recurrent case we know that no such limiting distribution can exist, since there is no finite invariant measure.

It is therefore remarkable that we can give a strong result on the closeness of the n-step distributions from different initial laws, even for chains which may be null.

Theorem 18.1.2 *Suppose $\boldsymbol{\Phi}$ is an aperiodic Harris recurrent chain. Then for any initial probability distributions λ, μ,*

$$\int\int \lambda(dx)\mu(dy)\|P^n(x,\,\cdot\,) - P^n(y,\,\cdot\,)\| \to 0, \qquad n \to \infty. \tag{18.13}$$

Proof Yet again we begin with the assumption that there is an atom $\boldsymbol{\alpha}$ in the space. Then for any x we have from the Regenerative Decomposition (13.47)

$$\|P^n(x,\,\cdot\,) - P^n(\boldsymbol{\alpha},\,\cdot\,)\| \le \mathsf{P}_x(\tau_\alpha \ge n) + |a_x * u - u|\,(n) + |a_x * u - u| * \overline{p}\,(n) \tag{18.14}$$

where now $\overline{p}(n) = \mathsf{P}_\alpha(\tau_\alpha > n)$. From Theorem 18.1.1 we know the last two terms in (18.14) tend to zero, whilst the first tends to zero from Harris recurrence.

The result (18.13) then follows for any two specific initial starting points x, y from the triangle inequality; it extends immediately to general initial distributions λ, μ from dominated convergence.

As previously, the extension to strongly aperiodic chains is straightforward, whilst the extension to general aperiodic chains follows from the contraction property of the total variation norm. □

We conclude with a consequence of this theorem which gives a uniform version of the fact that, in the null recurrent case, we have convergence of the transition probabilities to zero.

Theorem 18.1.3 *Suppose that $\boldsymbol{\Phi}$ is aperiodic and null recurrent, with invariant measure π. Then for any initial distribution λ and any constant $\varepsilon > 0$*

$$\lim_{n\to\infty} \sup_{A \in \mathcal{B}(\mathsf{X})} \int \lambda(dx) P^n(x, A)/[\pi(A) + \varepsilon] = 0. \tag{18.15}$$

PROOF Suppose by way of contradiction that we have a sequence of integers $\{n_k\}$ with $n_k \to \infty$ and a sequence of sets $B_k \in \mathcal{B}(\mathsf{X})$ such that, for some λ, and some $\delta, \varepsilon > 0$,

$$\int \lambda(dx) P^{n_k}(x, B_k) \geq \delta[\pi(B_k) + \varepsilon], \qquad k \in \mathbb{Z}_+. \tag{18.16}$$

Now from (18.13), we know that for every y

$$|\int \lambda(dx) P^{n_k}(x, B_k) - P^{n_k}(y, B_k)| \to 0, \qquad k \to \infty \tag{18.17}$$

and by Egorov's Theorem and the fact that $\pi(\mathsf{X}) = \infty$ this convergence is uniform on a set with π-measure arbitrarily large.

In particular we can take k and D such that $\pi(D) > \delta^{-1}$ and

$$|\int \lambda(dx) P^{n_k}(x, B_k) - P^{n_k}(y, B_k)| \leq \varepsilon\delta/2, \qquad y \in D. \tag{18.18}$$

Combining (18.16) and (18.18) gives

$$\begin{aligned} \pi(B_k) &= \int \pi(dy) P^{n_k}(y, B_k) \\ &\geq \int_D \pi(dy) P^{n_k}(y, B_k) \\ &\geq \pi(D)[\int \lambda(dx) P^{n_k}(x, B_k) - \varepsilon\delta/2] \\ &\geq \pi(D)[\delta(\pi(B_k) + \varepsilon) - \varepsilon\delta/2] \end{aligned} \tag{18.19}$$

which gives

$$\pi(D) \leq \delta^{-1},$$

thus contradicting the definition of D. □

The two results in Theorem 18.1.2 and Theorem 18.1.3 combine to tell us that, on the one hand, the distributions of the chain are getting closer as n gets large; and that they are getting closer on sets increasingly remote from the "center" of the space, as described by sets of finite π-measure.

18.2 Characterizing positivity using P^n

We have chosen to formulate positive recurrence initially, in Chapter 10, in terms of the finiteness of the invariant measure π. The ergodic properties of such chains are demonstrated in Chapters 13-16 as a consequence of this simple definition.

In contrast to this definition, the classical approach to the classification of irreducible chains as positive or null recurrent uses the transition probabilities rather than the invariant measure: typically, the invariant measure is demonstrated to exist only after a null/positive dichotomy is established in terms of the convergence properties of $P^n(x, A)$ for appropriate sets A. Null chains in this approach are those for which $P^n(x, A) \to 0$ for, say, all x and all small sets A, and almost by default, positive recurrent chains are those which are not null; that is, for which $\limsup P^n(x, A) > 0$.

We now develop a classification of states or of sets as positive recurrent or null using transition probabilities, and show that this approach is consistent with the definitions involving invariant measures in the case of ψ-irreducible chains.

18.2.1 Countable spaces

We will first consider the classical classification of null and positive chains based on P^n in the countable state space case.

When X is countable, recall that recurrence of individual states $x, y \in \mathsf{X}$ involves consideration of the finiteness or otherwise of $\mathsf{E}_x(\eta_y) = U(x,y) = \sum_{n=1}^{\infty} P^n(x,y)$. The stronger condition

$$\limsup_{n\to\infty} P^n(x,y) > 0 \tag{18.20}$$

obviously implies that

$$\mathsf{E}_x(\eta_y) = \infty; \tag{18.21}$$

and since in general, because of the cyclic behavior in Section 5.4, we may have

$$\liminf_{n\to\infty} P^n(x,y) = 0, \tag{18.22}$$

the condition (18.20) is often adopted as the next strongest stability condition after (18.21).

This motivates the following definitions.

> Null and positive states
>
> **(i)** The state $\boldsymbol{\alpha}$ is called *null* if $\lim_{n\to\infty} P^n(\boldsymbol{\alpha},\boldsymbol{\alpha}) = 0$.
>
> **(ii)** The state $\boldsymbol{\alpha}$ is called *positive* if $\limsup_{n\to\infty} P^n(\boldsymbol{\alpha},\boldsymbol{\alpha}) > 0$.

When $\boldsymbol{\Phi}$ is irreducible, either all states are positive or all states are null, since for any w, z there exist r, s such that $P^r(w,x) > 0$ and $P^s(y,z) > 0$ and

$$\limsup_{n\to\infty} P^{r+s+n}(w,z) > P^r(w,x)[\limsup_{n\to\infty} P^n(x,y)]P^s(y,z). \tag{18.23}$$

We need to show that these solidarity properties characterize positive and null chains in the sense we have defined them. One direction of this is easy, for if the chain is positive recurrent, with invariant probability π, then we have for any n

$$\pi(y) = \sum_x \pi(x) P^n(x,y);$$

hence if $\lim_{n\to\infty} P^n(w,w) = 0$ for some w then by (18.23) and dominated convergence $\pi(y) \equiv 0$, which is impossible. The other direction is easy only if one knows, not merely that $\limsup_{n\to\infty} P^n(x,y) > 0$ but that (at least through an aperiodic class) this is actually a limit. Theorem 18.1.3 now gives this to us.

Theorem 18.2.1 *If $\boldsymbol{\Phi}$ is irreducible on a countable space then the chain is positive recurrent if and only if some one state is positive. When $\boldsymbol{\Phi}$ is positive recurrent, for some $d \geq 1$*

$$\lim_{n\to\infty} P^{nd+r}(x,y) = d\pi(y) > 0$$

for all $x, y \in \mathsf{X}$, *and some* $0 \le r(x,y) \le d-1$; *and when* $\boldsymbol{\Phi}$ *is null*

$$\lim_{n\to\infty} P^{n}(x,y) = 0$$

for all $x, y \in \mathsf{X}$.

PROOF If the chain is transient then since $U(x,y) < \infty$ for all x, y from Proposition 8.1.1 we have that every state is null; whilst if the chain is null recurrent then since $\pi(y) < \infty$ for all y, Theorem 18.1.3 shows that every state is null.

Suppose that the chain is positive recurrent, with period d: then the Aperiodic Ergodic Theorem for the chain on the cyclic class D_j shows that for $x, y \in D_j$ we have

$$\lim_{n\to\infty} P^{nr}(x,y) = d\pi(y) > 0$$

whilst for $z \in D_{j-r \pmod d}$ we have $P^{j-r}(z, D_j) = 1$, showing that every state is positive. □

The simple equivalences in this result are in fact surprisingly hard to prove until we have established, not just the properties of the sequences $\limsup P^n$, but the actual existence of the limits of the sequences P^n through the periodic classes. This is why this somewhat elementary result has been reserved until now to establish.

18.2.2 General spaces

We now move on to the equivalent concepts for general chains: here, we must consider properties of sets rather than individual states, but we will see that the results above have completely general analogues.

When X is general, the definitions for sets which we shall use are

Null and positive sets

(i) The set A is called *null* if $\lim_{n\to\infty} P^n(x,A) = 0$ for all $x \in A$.

(ii) The set A is called *positive* if $\limsup_{n\to\infty} P^n(x,A) > 0$ for all $x \in A$.

We now prove that these definitions are consistent with the definitions of null and positive recurrence for general ψ-irreducible chains.

Theorem 18.2.2 *Suppose that* $\boldsymbol{\Phi}$ *is* ψ*-irreducible. Then*

(i) *the chain* $\boldsymbol{\Phi}$ *is positive recurrent if and only if every set* $B \in \mathcal{B}^+(\mathsf{X})$ *is positive;*

(ii) *if* $\boldsymbol{\Phi}$ *is null then every petite set is null and hence there is a sequence of null sets* B_j *with* $\bigcup_j B_j = \mathsf{X}$.

PROOF If the chain is null then either it is transient, in which case each petite set is strongly transient and thus null by Theorem 8.3.5; or it is null and recurrent in which case, since π exists and is finite on petite sets by Proposition 10.1.2, we have that every petite set is again null from Theorem 18.1.3.

Suppose the chain is positive recurrent and we have $A \in \mathcal{B}^+(\mathsf{X})$. For $x \in D_0 \cap H$, where H is the maximal Harris set, and D_0 is an arbitrary cyclic set, we have for each r

$$\lim_{n\to\infty} P^{nd+r}(x, A) = d\pi(A \cap D_r)$$

which is positive for some r. Since for every x we have $L(x, D_0 \cap H) > 0$ we have that the set A is positive. □

18.3 Positivity and T-chains

18.3.1 T-chains bounded in probability

In Chapter 12 we showed that chains on a topological space which are bounded in probability admit finite subinvariant measures under a wide range of continuity conditions.

It is thus reasonable to hope that ψ-irreducible chains on a topological space which are bounded in probability will be positive recurrent. Not surprisingly, we will see in this section that such a result is true for T-chains, and indeed we can say considerably more: boundedness in probability is actually equivalent to positive Harris recurrence in this case. Moreover, for T-chains positive or null sets also govern the behavior of the whole chain.

It is easy to see that on a countable space, where the continuous component properties are always satisfied, irreducible chains admit the following connection between boundedness in probability and positive recurrence.

Proposition 18.3.1 *For an irreducible chain on a countable space, positive Harris recurrence is equivalent to boundedness in probability.*

PROOF In the null case we do not have boundedness in probability since $P^n(x, y) \to 0$ for all x, y from Theorem 18.2.1.

In the positive case we have on each periodic set D_r a finite probability measure π_r such that if $x \in D_0$

$$\lim_{n\to\infty} P^{nd+r}(x, C) = \pi_r(C) \tag{18.24}$$

so by choosing a finite C such that $\pi_r(C) > 1-\varepsilon$ for all $1 \le r \le d$ we have boundedness in probability as required. □

The identical conclusion holds for T-chains. To get the broadest presentation, recall that a state $x^\star \in \mathsf{X}$ is *reachable* if

$$U(y, O) > 0$$

for every state $y \in \mathsf{X}$, and every open set O containing $x^\star$.

Theorem 18.3.2 *Suppose that $\boldsymbol{\Phi}$ is a T-chain and admits a reachable state $x^\star$. Then $\boldsymbol{\Phi}$ is a positive Harris chain if and only if it is bounded in probability.*

PROOF First note from Proposition 6.2.1 that for a T-chain the existence of just one reachable state $x^\star$ gives ψ-irreducibility, and thus $\boldsymbol{\Phi}$ is either positive or null.

Suppose that $\boldsymbol{\Phi}$ is bounded in probability. Then $\boldsymbol{\Phi}$ is non-evanescent from Proposition 12.1.1, and hence Harris recurrent from Theorem 9.2.2.

Moreover, boundedness in probability implies by definition that some compact set is non-null, and hence from Theorem 18.2.2 the chain is positive Harris, since compact sets are petite for T-chains.

Conversely, assume that the chain is positive Harris, with periodic sets D_j each supporting a finite probability measure π_j satisfying (18.24). Choose $\varepsilon > 0$, and compact sets $C_r \subseteq D_r$ such that $\pi_r(C_r) > 1 - \varepsilon$ for each r.

If $x \in D_j$ then with $C := \cup C_r$,

$$\lim_{n\to\infty} P^{nd+r-j}(x, C) = \pi_r(C_r) > 1 - \varepsilon. \tag{18.25}$$

If x is in the non-cyclic set $N = \mathsf{X} \setminus \cup D_j$ then $P^n(x, \cup D_j) \to 1$ by Harris recurrence, and thus from (18.25) we also have $\liminf_n P^n(x, C) > 1 - \varepsilon$, and this establishes boundedness in probability as required. □

18.3.2 Positive and null states for T-chains

The ideas encapsulated in the definitions of positive and null states in the countable case and positive and null sets in the general state space case find their counterparts in the local behavior of chains on spaces with a topology.

Analogously to the definition of topological recurrence at a point we have

> Topological positive and null recurrence of states
>
> We shall call a state x^*
>
> **(i)** *null* if $\lim_{n\to\infty} P^n(x^*, O) = 0$ for some neighborhood O of x^*;
>
> **(ii)** *positive* if $\limsup_{n\to\infty} P^n(x^*, O) > 0$ for all neighborhoods O of x^*.

We now show that these topological properties for points can be linked to their counterparts for the whole chain when the T-chain condition holds. This completes the series of results begun in Theorem 9.3.3 connecting global properties of T-chains with those at individual points.

Proposition 18.3.3 *Suppose that $\boldsymbol{\Phi}$ is a T-chain, and suppose that x^* is a reachable state. Then the chain $\boldsymbol{\Phi}$ is positive recurrent if and only if x^* is positive.*

PROOF From Proposition 6.2.1 the existence of a reachable state ensures the chain is ψ-irreducible. Assume that x^* is positive. Since $\boldsymbol{\Phi}$ is a T-chain, there exists an open petite set C containing x^* (take any precompact open neighborhood) and hence by Theorem 18.2.2 the chain is also positive.

Conversely, suppose that $\boldsymbol{\Phi}$ has an invariant probability π so that $\boldsymbol{\Phi}$ is positive recurrent. Since $x^\star$ is reachable it also lies in the support of π, and consequently any neighborhood of x^* is in $\mathcal{B}^+(\mathsf{X})$. Hence $x^\star$ is positive as required, from Theorem 18.2.2. □

18.4 Positivity and e-Chains

For T-chains we have a great degree of coherence in the concepts of positivity. Although there is not quite the same consistency for weak Feller chains, within the context of chains bounded in probability we can develop several valuable approaches, as we saw in Chapter 12.

In particular, for e-chains we now prove several further positivity results to indicate the level of work needed in the absence of ψ-irreducibility. It is interesting to note that it is the existence of a reachable state that essentially takes over the role of ψ-irreducibility, and that such states then interact well with the e-chain assumption.

18.4.1 Reachability and positivity

To begin we show that for an e-chain which is non-evanescent, the topological irreducibility condition that a reachable state exists is equivalent to the measure-theoretic irreducibility condition that the limiting measure $\Pi(x, \mathsf{X})$ is independent of the starting state x. Boundedness in probability on average is then equivalent to positivity of the reachable state.

We first give a general result for Feller chains:

Lemma 18.4.1 *If $\boldsymbol{\Phi}$ is a Feller chain and if a reachable state $x^\star$ exists, then for any pre-compact neighborhood O containing $x^\star$,*

$$\{\boldsymbol{\Phi} \to \infty\} = \{\boldsymbol{\Phi} \in O \text{ i.o.}\}^c \qquad \text{a.s. } [\mathsf{P}_*]$$

Proof Since $L(x, O)$ is a lower semicontinuous function of x by Proposition 6.1.1, and since by reachability it is strictly positive everywhere, it follows that $L(x, O)$ is bounded from below on compact subsets of X.

Letting $\{O_n\}$ denote a sequence of pre-compact open subsets of X with $O_n \uparrow \mathsf{X}$, it follows that $O_n \rightsquigarrow O$ for each n, and hence by Theorem 9.1.3 we have

$$\{\boldsymbol{\Phi} \in O_n \text{ i.o.}\} \subseteq \{\boldsymbol{\Phi} \in O \text{ i.o.}\} \qquad \text{a.s. } [\mathsf{P}_*]$$

This immediately implies that

$$\{\boldsymbol{\Phi} \to \infty\}^c = \bigcup_{n \geq 1} \{\boldsymbol{\Phi} \in O_n \text{ i.o.}\} \subseteq \{\boldsymbol{\Phi} \in O \text{ i.o.}\} \qquad \text{a.s. } [\mathsf{P}_*],$$

and since it is obvious that $\{\boldsymbol{\Phi} \to \infty\} \subseteq \{\boldsymbol{\Phi} \in O \text{ i.o.}\}^c$, this proves the lemma. □

Proposition 18.4.2 *Suppose that $\boldsymbol{\Phi}$ is an e-chain which is non-evanescent, and suppose that a reachable state $x^\star \in \mathsf{X}$ exists. Then the following are equivalent:*

(i) *There exists a unique invariant probability* π *such that*

$$\overline{P}_k(x,\,\cdot\,) \xrightarrow{\text{w}} \pi \qquad \text{as } k \to \infty;$$

(ii) $\boldsymbol{\Phi}$ *is bounded in probability on average;*

(iii) $x^\star$ *is positive.*

PROOF The identity $P\Pi = \Pi$ which is proved in Theorem 12.4.1 implies that for any $f \in \mathcal{C}_c(\mathsf{X})$, the adapted process $(\Pi(\Phi_k, f), \mathcal{F}_k^\Phi)$ is a bounded martingale. Hence by the Martingale Convergence Theorem D.6.1 there exists a random variable $\tilde{\pi}(f)$ for which

$$\lim_{k\to\infty} \Pi(\Phi_k, f) = \tilde{\pi}(f) \qquad \text{a.s. } [\mathsf{P}_*],$$

with $\mathsf{E}_y[\tilde{\pi}(f)] = \Pi(y,f)$ for all $y \in \mathsf{X}$.

Since $\Pi(y,f)$ is a continuous function of y, it follows from Lemma 18.4.1 that

$$\liminf_{k\to\infty} |\Pi(\Phi_k, f) - \Pi(x^\star, f)| = 0 \qquad \text{a.s. } [\mathsf{P}_*],$$

which gives $\tilde{\pi}(f) = \Pi(x^\star, f)$ a.s. $[\mathsf{P}_*]$. Taking expectations gives $\Pi(y,f) = \mathsf{E}_y[\tilde{\pi}(f)] = \Pi(x^\star, f)$ for all y.

Since a finite measure on $\mathcal{B}(\mathsf{X})$ is determined by its values on continuous functions with compact support, this shows that the measures $\Pi(y,\,\cdot\,)$, $y \in \mathsf{X}$, are identical. Let π denote their common value.

To prove Proposition 18.4.2 we first show that (i) and (iii) are equivalent. To see that (iii) implies (i), observe that under positivity of $x^\star$ we have $\Pi(x^\star, \mathsf{X}) > 0$, and since $\Pi(y, \mathsf{X}) = \pi(\mathsf{X})$ does not depend on y it follows from Theorem 12.4.3 that $\Pi(y, \mathsf{X}) = 1$ for all y. Hence π is an invariant probability, which shows that (i) does hold.

Conversely, if (i) holds then by reachability of $x^\star$ we have $x^\star \in \operatorname{supp} \pi$ and hence every neighborhood of $x^\star$ is positive. This shows that (iii) also holds.

We now show that (i) is equivalent to (ii).

It is obvious that (i) implies (ii). To see the converse, observe that if (ii) holds then by Theorem 12.4.1 we have that π is an invariant probability. Moreover, since $x^\star$ is reachable we must have that $\pi(O) > 0$ for any neighborhood of $x^\star$. Since $\Pi(y, O) = \pi(O)$ for every y, this shows that $x^\star$ is positive.

Hence (iii) holds, which implies that (i) also holds. □

18.4.2 Aperiodicity and convergence

The existence of a limit for $\overline{P}_k$ in Proposition 18.4.2 rather than for the individual terms P^n seems to follow naturally in the topology we are using here.

We can strengthen such convergence results using a topological notion of aperiodicity and we turn to such concepts in the this section. It appears to be a particularly difficult problem to find such limits for the terms P^n in the weak Feller situation without an e-chain condition.

In the topological case we use a definition justified by the result in Lemma D.7.4, which is one of the crucial consequences of the definitions in Chapter 5.

Topological aperiodicity of states

A recurrent state x is called *aperiodic* if $P^k(x, O) > 0$ for each open set O containing x, and all $k \in \mathbb{Z}_+$ sufficiently large.

The following result justifies this definition of aperiodicity and strengthens Theorem 12.4.1.

Proposition 18.4.3 *Suppose that Φ is an e-chain which is bounded in probability on average. Let $x^\star \in \mathsf{X}$ be reachable and aperiodic, and let $\pi = \Pi(x^\star, \cdot\,)$. Then for each initial condition y lying in* $\operatorname{supp}\pi$,

$$P^k(y, \cdot\,) \xrightarrow{\mathrm{w}} \pi \qquad \text{as } k \to \infty \tag{18.26}$$

PROOF For any $f \in \mathcal{C}_c(\mathsf{X})$ we have by stationarity,

$$\int |P^k f| \, d\pi = \int [\int P|P^k f|\,] d\pi \geq \int |P^{k+1} f| \, d\pi,$$

and hence $v := \lim_{k\to\infty} \int |P^k f| \, d\pi$ exists.

Since $\{P^k f\}$ is equicontinuous on compact subsets of X, there exists a continuous function g, and a subsequence $\{k_i\} \subset \mathbb{Z}_+$ for which $P^{k_i} f \to g$ as $i \to \infty$ uniformly on compact subsets of X. Hence we also have $P^{k_i+\ell} f \to P^\ell g$ as $i \to \infty$ uniformly on compact subsets of X.

By the Dominated Convergence Theorem we have for all $\ell \in \mathbb{Z}_+$,

$$\int P^\ell g \, d\pi = \int f \, d\pi \quad \text{and} \quad \int |P^\ell g| \, d\pi = v. \tag{18.27}$$

We will now show that this implies that the function g cannot change signs on $\operatorname{supp}\pi$.

Suppose otherwise, so that the open sets

$$O_+ := \{x \in \mathsf{X} : g(x) > 0\}, \qquad O_- := \{x \in \mathsf{X} : g(x) < 0\}$$

both have positive π measure.

Because $x^\star \in \operatorname{supp}\pi$, it follows by Proposition 18.4.2 that there exists $k_+,\ k_- \in \mathbb{Z}_+$ such that

$$P^{k_+}(y, O_+) > 0 \quad \text{and} \quad P^{k_-}(y, O_-) > 0 \tag{18.28}$$

when $y = x^\star$, and since $P^n(\,\cdot\,, O)$ is lower semicontinuous for any open set $O \subset \mathsf{X}$, equation (18.28) holds for all y in an open neighborhood N containing $x^\star$.

We may now use aperiodicity. Since $P^k(x^\star, N) > 0$ for all k sufficiently large, we deduce from (18.28) that there exists $\ell \in \mathbb{Z}_+$ for which

$$P^\ell(y, O_+) > 0 \qquad \text{and} \quad P^\ell(y, O_-) > 0$$

when $y = x^\star$, and hence for all y in an open neighborhood N' of $x^\star$. This implies that $|P^\ell g| < P^\ell |g|$ on N', and since $\pi\{N'\} > 0$, that $\int |P^\ell g| \, d\pi < \int |g| \, d\pi$, in contradiction to the second equality in (18.27).

Hence g does not change signs in $\operatorname{supp}\pi$. But by (18.27) it follows that if $\int f\,d\pi = 0$ then

$$0 = |\int g\,d\pi| = \int |g|\,d\pi,$$

so that $g \equiv 0$ on $\operatorname{supp}\pi$. This shows that the limit (18.26) holds for all initial conditions in $\operatorname{supp}\pi$. □

We now show that if a reachable state exists for an e-chain then the limit in Proposition 18.4.3 holds for each initial condition. A sample path version of Theorem 18.4.4 will be presented below.

Theorem 18.4.4 *Suppose that $\boldsymbol{\Phi}$ is an e-chain which is bounded in probability on average. Then*

(i) *A unique invariant probability π exists if and only if a reachable state $x^\star \in \mathsf{X}$ exists;*

(ii) *If an aperiodic reachable state $x^\star \in \mathsf{X}$ exists, then for each initial state $x \in \mathsf{X}$,*

$$P^k(x, \cdot\,) \xrightarrow{\mathrm{w}} \pi \qquad \text{as } k \to \infty, \tag{18.29}$$

where π is the unique invariant probability for $\boldsymbol{\Phi}$. Conversely, if (18.29) holds for all $x \in \mathsf{X}$ then every state in $\operatorname{supp}\pi$ *is reachable and aperiodic.*

Proof The proof of (i) follows immediately from Proposition 18.4.2, and the converse of (ii) is straightforward.

To prove the remainder, we assume that the state $x^\star \in \mathsf{X}$ is reachable and aperiodic, and show that equation (18.29) holds for all initial conditions.

Suppose that $\int f\,d\pi = 0$, $|f(x)| \leq 1$ for all x, and for fixed $\varepsilon > 0$ define the set

$$O_\varepsilon := \{x \in \mathsf{X} : \limsup_{k\to\infty} |P^k f| < \varepsilon\}.$$

Because the Markov transition function P is equicontinuous, and because Proposition 18.4.3 implies that (18.29) holds for all initial conditions in $\operatorname{supp}\pi$, the set O_ε is an open neighborhood of $\operatorname{supp}\pi$.

Hence $\pi\{O_\varepsilon\} = 1$, and since O_ε is open, it follows from Theorem 18.4.4 (i) that

$$\lim_{N\to\infty} \overline{P}_N(x, O_\varepsilon) = 1.$$

Fix $x \in \mathsf{X}$, and choose $N_0 \in \mathbb{Z}_+$ such that $P^{N_0}(x, O_\varepsilon) \geq 1 - \varepsilon$. We then have by the definition of O_ε and Fatou's Lemma,

$$\begin{aligned}\limsup_{k\to\infty} |P^{N_0+k} f\,(x)| &\leq P^{N_0}(x, O_\varepsilon^c) + \limsup_{k\to\infty} \int_{O_\varepsilon} P^{N_0}(x, dy)|P^k f\,(y)| \\ &\leq 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, this completes the proof. □

18.5 The LLN for e-Chains

As a final result, illustrating both these methods and the sample path methods developed in Chapter 17, we now give a sample path version of Proposition 18.4.2 for e-chains.

Define the *occupation probabilities* as

$$\tilde{\mu}_n\{A\} := S_n(\mathbb{1}_A) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}\{\Phi_k \in A\} \qquad n \in \mathbb{Z}_+, \quad A \in \mathcal{B}(\mathsf{X}). \tag{18.30}$$

Observe that $\{\tilde{\mu}_k\}$ are not probabilities in the usual sense, but are probability-valued random variables.

The Law of Large Numbers (Theorem 17.1.2) states that if an invariant probability measure π exists, then the occupation probabilities converge with probability one for each initial condition lying in a set of full π-measure. We now present two versions of the law of large numbers for e-chains where the null set appearing in Theorem 17.1.2 is removed by restricting consideration to continuous, bounded functions. The first is a Weak Law of Large Numbers, since the convergence is only in probability, while the second is a Strong Law with convergence occurring almost surely.

Theorem 18.5.1 *Suppose that $\boldsymbol{\Phi}$ is an e-chain bounded in probability on average, and suppose that a reachable state exists. Then a unique invariant probability π exists and the following limits hold.*

(i) *For any $f \in \mathcal{C}(\mathsf{X})$, as $k \to \infty$*

$$\int f\, d\tilde{\mu}_k \to \int f\, d\pi$$

in probability for each initial condition;

(ii) *If for each initial condition of the Markov chain the occupation probabilities are almost surely tight, then as $k \to \infty$*

$$\tilde{\mu}_k \xrightarrow{\mathrm{w}} \pi \qquad \text{a.s. } [\mathsf{P}_*]. \tag{18.31}$$

PROOF Let $f \in \mathcal{C}(\mathsf{X})$ with $0 \le f(x) \le 1$ for all x, let $C \subset \mathsf{X}$ be compact and choose $\varepsilon > 0$. Since $\overline{P}_k f \to \int f\, d\pi$ as $k \to \infty$, uniformly on compact subsets of X, there exists M sufficiently large for which

$$\Big|\frac{1}{N}\sum_{k=1}^{N} \overline{P}_M f\,(\Phi_k) - \int f\, d\pi\Big| \le \varepsilon + \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}\{\Phi_i \in C^c\} \tag{18.32}$$

Now for any $M \in \mathbb{Z}_+$, we will show

$$|\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k) - \int f\, d\pi| = |\frac{1}{N}\sum_{k=1}^{N} \overline{P}_M f(\Phi_k) - \int f\, d\pi| + o(1) \tag{18.33}$$

where the term $o(1)$ converges to zero as $n \to \infty$ with probability one.

For each N, $n \in \mathbb{Z}_+$ we have

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k) - \int f\,d\pi \quad &= \quad \sum_{i=0}^{n-1}\frac{1}{N}\sum_{k=1}^{N}\Big(P^i f(\Phi_{k-i}) - P^{i+1} f(\Phi_{k-i-1})\Big) \\ &\quad + \frac{1}{N}\sum_{k=1}^{N} P^n f(\Phi_k) - \int f\,d\pi \\ &\quad + \frac{1}{N}\sum_{k=1}^{N}\Big(P^n f(\Phi_{k-n}) - P^n f(\Phi_k)\Big)\end{aligned}$$

where we adopt the convention that $\Phi_k = \Phi_0$ for $k \le 0$. For each $M \in \mathbb{Z}_+$ we may average the right hand side of this equality from $n = 1$ to M to obtain

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k) - \int f\,d\pi \quad &= \quad \frac{1}{M}\sum_{n=1}^{M}\left(\sum_{i=0}^{n-1}\frac{1}{N}\sum_{k=1}^{N}\Big(P^i f(\Phi_{k-i}) - P^{i+1} f(\Phi_{k-i-1})\Big)\right) \\ &\quad + \frac{1}{N}\sum_{k=1}^{N}\Big(\frac{1}{M}\sum_{n=1}^{M} P^n f(\Phi_k)\Big) - \int f\,d\pi \\ &\quad + \frac{1}{M}\sum_{n=1}^{M}\Big(\frac{1}{N}\sum_{k=1}^{N} P^n f(\Phi_{k-n}) - P^n f(\Phi_k)\Big)\end{aligned}$$

The fourth term is a telescoping series, and hence recalling our definition of the transition function $\overline{P}_M$ we have

$$\begin{aligned}\Big|\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k) - \int f\,d\pi\Big| \quad &\le \quad \sum_{i=0}^{M-1}\Big|\frac{1}{N}\sum_{k=1}^{N}\Big(P^i f(\Phi_{k-i}) - P^{i+1} f(\Phi_{k-i-1})\Big)\Big| \\ &\quad + \Big|\frac{1}{N}\sum_{k=1}^{N}\Big(\overline{P}_M f\,(\Phi_k) - \int f\,d\pi\Big)\Big| \\ &\quad + \frac{2M}{N} \qquad (18.34)\end{aligned}$$

For each fixed $0 \le i \le M-1$ the sequence

$$\Big(P^i f(\Phi_{k-i}) - P^{i+1} f(\Phi_{k-i-1}), \mathcal{F}^{\Phi}_{k-i}\Big) \qquad k > i,$$

is a bounded martingale difference process. Hence by Theorem 5.2 of Chapter 4 of [68], the first summand converges to zero almost surely for every $M \in \mathbb{Z}_+$, and thus (18.33) is proved.

Hence for any $\gamma > \varepsilon$, it follows from (18.33) and (18.32) that

$$\begin{aligned}&\limsup_{N\to\infty} \mathsf{P}_x\Big\{\Big|\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k) - \int f\,d\pi\Big| \ge \gamma\Big\} \\ &\quad \le \quad \limsup_{N\to\infty} \mathsf{P}_x\Big\{\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}\{\Phi_i \in C^c\} \ge \gamma - \varepsilon\Big\} \\ &\quad \le \quad \frac{1}{\gamma-\varepsilon}\limsup_{N\to\infty} \mathsf{E}_x\Big[\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}\{\Phi_i \in C^c\}\Big]\end{aligned}$$

Since $\boldsymbol{\Phi}$ is bounded in probability on average, the right hand side decreases to zero as $C \uparrow \mathsf{X}$, which completes the proof of (i).

To prove (ii), suppose that the occupation probabilities $\{\tilde{\mu}_k\}$ are tight along some sample path. Then we may choose the compact set C in (18.32) so that along this sample path

$$\limsup_{N\to\infty}\Big|\frac{1}{N}\sum_{k=1}^{N}\overline{P}_M f\,(\Phi_k) - \int f\,d\pi\Big| \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (18.33) shows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f\,(\Phi_k) = \int f\,d\pi \qquad \text{a.s. } [\mathsf{P}_*]$$

so that the Strong Law of Large Numbers holds for all $f \in \mathcal{C}(\mathsf{X})$ and all initial conditions $x \in \mathsf{X}$.

Let $\{f_n\}$ be a sequence of continuous functions with compact support which is dense in $\mathcal{C}_c(\mathsf{X})$ in the uniform norm. Such a sequence exists by Proposition D.5.1. Then by the preceding result,

$$\mathsf{P}_x\Big\{\lim_{k\to\infty}\int f_n d\tilde{\mu}_k = \int f_n d\pi \qquad \text{for each } n \in \mathbb{Z}_+\Big\} = 1,$$

which implies that $\tilde{\mu}_k \xrightarrow{\text{v}} \pi$ as $k \to \infty$. Since π is a probability, this shows that in fact $\tilde{\mu}_k \xrightarrow{\text{w}} \pi$ a.s. $[\mathsf{P}_*]$, and this completes the proof. □

We conclude by stating a result which, combined with Theorem 18.5.1, provides a test function approach to establishing the Law of Large Numbers for $\boldsymbol{\Phi}$. For a proof see [169].

Theorem 18.5.2 *If a norm-like function V and a compact set C satisfy condition (V4) then $\boldsymbol{\Phi}$ is bounded in probability, and the occupation probabilities are almost surely tight for each initial condition. Hence, if $\boldsymbol{\Phi}$ is an e-chain, and if a reachable state exists,*

$$\tilde{\mu}_k \xrightarrow{\text{w}} \pi \quad \textit{as } k \to \infty \qquad \text{a.s. } [\mathsf{P}_*]. \tag{18.35}$$

18.6 Commentary

Theorem 18.1.2 for positive recurrent chains is first proved in Orey [207], and the null recurrent version we give here is in Jamison and Orey [111]. The dependent coupling which we use to prove this result for null recurrent chains is due to Ornstein [209], [210], and is also developed in Berbee [20]. Our presentation of this material has relied heavily on Nummelin [202], and further related results can be found in his Chapter 6.

Theorem 18.1.3 is due to Jain [105], and our proof is taken from Orey [208].

The links between positivity of states, boundedness in probability, and positive Harris recurrence for T-chains are taken from Meyn [169], Meyn and Tweedie [178] and Tuominen and Tweedie [269]. In [178] analogues of Theorem 18.3.2 and Proposition 18.3.3 are obtained for non-irreducible chains.

The convergence result Theorem 18.4.4 for chains possessing an aperiodic reachable state is based upon Theorem 8.7.2 of Feller [77].

The use of the martingale property of $\Pi(\Phi_k, f)$ to obtain uniqueness of the invariant probability in Proposition 18.4.2 is originally in [109]. This is a powerful technique which is perhaps even more interesting in the absence of a reachable state.

For suppose that the chain is bounded in probability but a reachable state does not exist, and define an equivalence relation on X as follows: $x \leftrightarrow y$ if and only if $\mathit{\Pi}(x, \cdot) = \mathit{\Pi}(y, \cdot)$. It follows from the same techniques which were used in the proof of Proposition 18.4.2, that if x is recurrent then the set of all states $\overline{E}_x$ for which $y \leftrightarrow x$ is closed. Since $x \in \overline{E}_x$ for every recurrent point $x \in R$, $F = \mathsf{X} - \sum \overline{E}_x$ consists entirely of non-recurrent points. It then follows from Proposition 3.3 of Tuominen and Tweedie [270] that F is transient.

From this decomposition and Proposition 18.4.3 it is straightforward to generalize Theorem 18.4.4 to chains which do not possess a reachable state. The details of this decomposition are spelled out in Meyn and Tweedie [182].

Such decompositions have a large literature for Feller chains and e-chains: see for example Jamison [109] and also Rosenblatt [227] for e-chains, and Jamison and Sine [112], Sine [243, 242, 241] and Foguel [78, 80] for Feller chains and the detailed connections between the Feller property and the stronger e-chain property. All of these papers consider exclusively compact state spaces. The results for non-compact state spaces appear here for the first time.

The LLN for e-chains is originally due to Breiman [29] who considered Feller chains on a compact state space. Also on a compact state space is Jamison's extension of Breiman's result [108] where the LLN is obtained without the assumption that a unique invariant probability exists.

One of the apparent difficulties in establishing this result is finding a candidate limit $\tilde{\pi}(f)$ of the sample path averages $\frac{1}{n}S_n(f)$. Jamison resolved this by considering the transition function $\mathit{\Pi}$, and the associated convergent martingale $(\mathit{\Pi}(\Phi_k, A), \mathcal{F}_k^{\Phi})$. If the chain is bounded in probability on average then we define the *random probability* $\tilde{\pi}$ as

$$\tilde{\pi}\{A\} := \lim_{k\to\infty} \mathit{\Pi}(\Phi_k, A), \qquad A \in \mathcal{B}(\mathsf{X}). \tag{18.36}$$

It is then easy to show by modifying (18.34) that Theorem 18.5.1 continues to hold with $\int f\, d\pi$ replaced by $\int f\, d\tilde{\pi}$, even when no reachable state exists for the chain. The proof of Theorem 18.5.1 can be adopted after it is appropriately modified using the limit (18.36).