9

Harris and Topological Recurrence

In this chapter we consider stronger concepts of recurrence and link them with the dichotomy proved in Chapter 8. We also consider several obvious definitions of global and local recurrence and transience for chains on topological spaces, and show that they also link to the fundamental dichotomy.

In developing concepts of recurrence for sets $A \in \mathcal{B}(\mathsf{X})$, we will consider not just the first hitting time τ_A, or the expected value $U(\,\cdot\,, A)$ of η_A, but also the event that $\boldsymbol{\Phi} \in A$ infinitely often (i.o.), or $\eta_A = \infty$, defined by

$$\{\boldsymbol{\Phi} \in A \text{ i.o.}\} := \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{\Phi_k \in A\}$$

which is well defined as an $\mathcal{F}$-measurable event on Ω. For $x \in \mathsf{X}$, $A \in \mathcal{B}(\mathsf{X})$ we write

$$Q(x, A) := \mathsf{P}_x\{\boldsymbol{\Phi} \in A \text{ i.o.}\} : \tag{9.1}$$

obviously, for any x, A we have $Q(x, A) \leq L(x, A)$, and by the strong Markov property we have

$$Q(x, A) = \mathsf{E}_x[\mathsf{P}_{\Phi_{\tau_A}}\{\boldsymbol{\Phi} \in A \text{ i.o.}\}\mathbb{1}\{\tau_A < \infty\}] = \int_A U_A(x, dy) Q(y, A). \tag{9.2}$$

Harris recurrence

The set A is called *Harris recurrent* if

$$Q(x, A) = \mathsf{P}_x(\eta_A = \infty) = 1, \qquad x \in A.$$

A chain $\boldsymbol{\Phi}$ is called *Harris (recurrent)* if it is ψ-irreducible and every set in $\mathcal{B}^+(\mathsf{X})$ is Harris recurrent.

We will see in Theorem 9.1.4 that when $A \in \mathcal{B}^+(\mathsf{X})$ and $\boldsymbol{\Phi}$ is Harris recurrent then in fact we have the seemingly stronger and perhaps more commonly used property that $Q(x, A) = 1$ for every $x \in \mathsf{X}$.

It is obvious from the definitions that if a set is Harris recurrent, then it is recurrent. Indeed, in the formulation above the strengthening from recurrence to Harris recurrence is quite explicit, indicating a move from an expected infinity of visits to an almost surely infinite number of visits to a set.

This definition of Harris recurrence appears on the face of it to be stronger than requiring $L(x, A) \equiv 1$ for $x \in A$, which is a standard alternative definition of Harris recurrence. In one of the key results of this section, Proposition 9.1.1, we prove that they are in fact equivalent.

The highlight of the Harris recurrence analysis is

Theorem 9.0.1 *If $\boldsymbol{\Phi}$ is recurrent, then we can write*

$$\mathsf{X} = H \cup N \tag{9.3}$$

where H is absorbing and non-empty and every subset of H in $\mathcal{B}^+(\mathsf{X})$ is Harris recurrent; and N is ψ-null and transient.

PROOF This is proved, in a slightly stronger form, in Theorem 9.1.5. □

Hence a recurrent chain differs only by a ψ-null set from a Harris recurrent chain. In general we can then restrict analysis to H and derive very much stronger results using properties of Harris recurrent chains.

For chains on a countable space the null set N in (9.3) is empty, so recurrent chains are automatically Harris recurrent.

On a topological space we can also find conditions for this set to be empty, and these also provide a useful interpretation of the Harris property.

We say that a sample path of $\boldsymbol{\Phi}$ *converges to infinity* (denoted $\boldsymbol{\Phi} \to \infty$) if the trajectory visits each compact set only finitely often. This definition leads to

Theorem 9.0.2 *For a ψ-irreducible T-chain, the chain is Harris recurrent if and only if $\mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\} = 0$ for each $x \in \mathsf{X}$.*

PROOF This is proved in Theorem 9.2.2 □

Even without its equivalence to Harris recurrence for such chains this "recurrence" type of property (which we will call *non-evanescence*) repays study, and this occupies Section 9.2.

In this chapter, we also connect local recurrence properties of a chain on a topological space with global properties: if the chain is a ψ-irreducible T-chain, then recurrence of the neighborhoods of any one point in the support of ψ implies recurrence of the whole chain.

Finally, we demonstrate further connections between drift conditions and Harris recurrence, and apply these results to give an increment analysis of chains on $\mathbb{R}$ which generalizes that for the random walk in the previous chapter.

9.1 Harris recurrence

9.1.1 Harris properties of sets

We first develop conditions to ensure that a set is Harris recurrent, based only on the first return time probabilities $L(x, A)$.

Proposition 9.1.1 *Suppose for some one set $A \in \mathcal{B}(\mathsf{X})$ we have $L(x, A) \equiv 1, x \in A$. Then $Q(x, A) = L(x, A)$ for every $x \in \mathsf{X}$, and in particular A is Harris recurrent.*

PROOF Using the strong Markov property, we have that if $L(y, A) = 1$, $y \in A$, then for any $x \in A$

$$\mathsf{P}_x(\tau_A(2) < \infty) = \int_A U_A(x, dy) L(y, A) = 1;$$

inductively this gives for $x \in A$, again using the strong Markov property,

$$\mathsf{P}_x(\tau_A(k+1) < \infty) = \int_A U_A(x, dy) \mathsf{P}_y(\tau_A(k) < \infty) = 1.$$

For any x we have

$$\mathsf{P}_x(\eta_A \geq k) = \mathsf{P}_x(\tau_A(k) < \infty),$$

and since by monotone convergence

$$Q(x, A) = \lim_k \mathsf{P}_x(\eta_A \geq k)$$

we have $Q(x, A) \equiv 1$ for $x \in A$.

It now follows since

$$Q(x, A) = \int_A U_A(x, dy) Q(y, A) = L(x, A)$$

that the theorem is proved. □

This shows that the definition of Harris recurrence in terms of Q is identical to a similar definition in terms of L: the latter is often used (see for example Orey [208]) but the use of Q highlights the difference between recurrence and Harris recurrence.

We illustrate immediately the usefulness of the stronger version of recurrence in conjunction with the basic dichotomy to give a proof of transience of random walk on $\mathbb{Z}$.

We showed in Section 8.4.3 that random walk on $\mathbb{Z}$ is transient when the increment has non-zero mean and the range of the increment is bounded.

Using the fact that, on the integers, recurrence and Harris recurrence are identical from Proposition 8.1.3, we can remove this bounded range restriction. To do this we use the strong rather than the weak law of large numbers, as used in Theorem 8.1.5.

The form we require (see again, for example, Billingsley [25]) states that if Φ_n is a random walk such that the increment distribution Γ has a mean β which is not zero, then

$$\mathsf{P}_0(\lim_{n \to \infty} n^{-1} \Phi_n = \beta) = 1.$$

Write C_n for the event $\{|n^{-1}\Phi_n - \beta| > \beta/2\}$. We only use the result, which follows from the strong law, that

$$\mathsf{P}_0(\limsup_{n\to\infty} C_n) = 0. \tag{9.4}$$

Now let D_n denote the event $\{\Phi_n = 0\}$, and notice that $D_n \subseteq C_n$ for each n. Immediately from (9.4) we have

$$\mathsf{P}_0(\limsup_{n\to\infty} D_n) = 0 \tag{9.5}$$

which says exactly $Q(0,0) = 0$.

Hence we have an elegant proof of the general result

Proposition 9.1.2 *If $\boldsymbol{\Phi}$ denotes random walk on $\mathbb{Z}$ and if*

$$\beta = \sum w\, \Gamma(w) > 0$$

then $\boldsymbol{\Phi}$ is transient. □

The most difficult of the results we prove in this section, and the strongest, provides a rather more delicate link between the probabilities $L(x,A)$ and $Q(x,A)$ than that in Proposition 9.1.1.

Theorem 9.1.3 **(i)** *Suppose that $D \rightsquigarrow A$ for any sets D and A in $\mathcal{B}(\mathsf{X})$. Then*

$$\{\boldsymbol{\Phi} \in D \text{ i.o.}\} \subseteq \{\boldsymbol{\Phi} \in A \text{ i.o.}\} \qquad a.s.\ [\mathsf{P}_*] \tag{9.6}$$

and hence $Q(y,D) \le Q(y,A)$, for all $y \in \mathsf{X}$.

(ii) *If $\mathsf{X} \rightsquigarrow A$ then A is Harris recurrent, and in fact $Q(x,A) \equiv 1$ for every $x \in \mathsf{X}$.*

Proof Since the event $\{\boldsymbol{\Phi} \in A \text{ i.o.}\}$ involves the whole path of $\boldsymbol{\Phi}$, we cannot deduce this result merely by considering P^n for fixed n. We need to consider all the events

$$E_n = \{\Phi_{n+1} \in A\}, \qquad n \in \mathbb{Z}_+$$

and evaluate the probability of those paths such that an infinite number of the E_n hold.

We first show that, if $\mathcal{F}_n^{\Phi}$ is the σ-field generated by $\{\Phi_0, \ldots, \Phi_n\}$, then as $n \to \infty$

$$\mathsf{P}\Big[\bigcup_{i=n}^{\infty} E_i \mid \mathcal{F}_n^{\Phi}\Big] \to \mathbb{1}\Big(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} E_i\Big) \qquad \text{a.s.} \quad [\mathsf{P}_*] \tag{9.7}$$

To see this, note that for fixed $k \le n$

$$\mathsf{P}\Big[\bigcup_{i=k}^{\infty} E_i \mid \mathcal{F}_n^{\Phi}\Big] \ge \mathsf{P}\Big[\bigcup_{i=n}^{\infty} E_i \mid \mathcal{F}_n^{\Phi}\Big] \ge \mathsf{P}\Big[\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} E_i \mid \mathcal{F}_n^{\Phi}\Big]. \tag{9.8}$$

Now apply the Martingale Convergence Theorem (see Theorem D.6.1) to the extreme elements of the inequalities (9.8) to give

$$\begin{aligned} \mathbb{1}\Big[\textstyle\bigcup_{i=k}^{\infty} E_i\Big] &\ge \limsup_n \mathsf{P}\Big[\textstyle\bigcup_{i=n}^{\infty} E_i \mid \mathcal{F}_n^{\Phi}\Big] \\ &\ge \liminf_n \mathsf{P}\Big[\textstyle\bigcup_{i=n}^{\infty} E_i \mid \mathcal{F}_n^{\Phi}\Big] \\ &\ge \mathbb{1}\Big[\textstyle\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} E_i\Big]. \end{aligned} \tag{9.9}$$

As $k \to \infty$, the two extreme terms in (9.9) converge, which shows the limit in (9.7) holds as required.

By the strong Markov property, $\mathsf{P}_*[\bigcup_{i=n}^\infty E_i \mid \mathcal{F}_n^\Phi] = L(\Phi_n, A)$ a.s. $[\mathsf{P}_*]$. From our assumption that $D \rightsquigarrow A$ we have that $L(\Phi_n, A)$ is bounded from 0 whenever $\Phi_n \in D$. Thus, using (9.7) we have P_*-a.s,

$$\begin{aligned} \mathbb{1}\Big(\bigcap_{m=1}^\infty \bigcup_{i=m}^\infty \{\Phi_i \in D\}\Big) &\leq \mathbb{1}\Big(\limsup_n L(\Phi_n, A) > 0\Big) \\ &= \mathbb{1}\Big(\lim_n L(\Phi_n, A) = 1\Big) \\ &= \mathbb{1}\Big(\bigcap_{m=1}^\infty \bigcup_{i=m}^\infty E_i\Big), \end{aligned} \tag{9.10}$$

which is (9.6).

The proof of (ii) is then immediate, by taking $D = \mathsf{X}$ in (9.6). □

As an easy consequence of Theorem 9.1.3 we have the following strengthening of Harris recurrence:

Theorem 9.1.4 *If $\boldsymbol{\Phi}$ is Harris recurrent then $Q(x, B) = 1$ for every $x \in \mathsf{X}$ and every $B \in \mathcal{B}^+(\mathsf{X})$.*

Proof Let $\{C_n : n \in \mathbb{Z}_+\}$ be petite sets with $\cup C_n = \mathsf{X}$. Since the finite union of petite sets is petite for an irreducible chain by Proposition 5.5.5, we may assume that $C_n \subset C_{n+1}$ and that $C_n \in \mathcal{B}^+(\mathsf{X})$ for each n.

For any $B \in \mathcal{B}^+(\mathsf{X})$ and any $n \in \mathbb{Z}_+$ we have from Lemma 5.5.1 that $C_n \rightsquigarrow B$, and hence, since C_n is Harris recurrent, we see from Theorem 9.1.3 (i) that $Q(x, B) = 1$ for any $x \in C_n$. Because the sets $\{C_k\}$ cover X, it follows that $Q(x, B) = 1$ for all x as claimed. □

Having established these stability concepts, and conditions implying they hold for individual sets, we now move on to consider transience and recurrence of the overall chain in the ψ-irreducible context.

9.1.2 Harris recurrent chains

It would clearly be desirable if, as in the countable space case, every set in $\mathcal{B}^+(\mathsf{X})$ were Harris recurrent for every recurrent $\boldsymbol{\Phi}$. Regrettably this is not quite true.

For consider any chain $\boldsymbol{\Phi}$ for which every set in $\mathcal{B}^+(\mathsf{X})$ is Harris recurrent: append to X a sequence of individual points $N = \{x_i\}$, and expand P to P' on $\mathsf{X}' := \mathsf{X} \cup N$ by setting $P'(x, A) = P(x, A)$ for $x \in \mathsf{X}, A \in \mathcal{B}(\mathsf{X})$, and

$$P'(x_i, x_{i+1}) = \beta_i, \qquad P'(x_i, \boldsymbol{\alpha}) = 1 - \beta_i$$

for some one specific $\boldsymbol{\alpha} \in \mathsf{X}$ and all $x_i \in N$.

Any choice of the probabilities β_i which provides

$$1 > \prod_{i=0}^\infty \beta_i > 0$$

then ensures that

$$L'(x_i, A) = L'(x_i, \boldsymbol{\alpha}) = 1 - \prod_{n=i}^\infty \beta_i < 1, \qquad A \in \mathcal{B}^+(\mathsf{X})$$

so that no set $B \subset \mathsf{X}'$ with $B \cap \mathsf{X}$ in $\mathcal{B}^+(\mathsf{X})$ and $B \cap N$ non-empty is Harris recurrent: but

$$U'(x_i, A) \geq L'(x_i, \boldsymbol{\alpha})U(\boldsymbol{\alpha}, A) = \infty, \qquad A \in \mathcal{B}(\mathsf{X})$$

so that every set in $\mathcal{B}^+(\mathsf{X}')$ is recurrent.

We now show that this example typifies the only way in which an irreducible chain can be recurrent and not Harris recurrent: that is, by the existence of an absorbing set which is Harris recurrent, accompanied by a single ψ-null set on which the Harris recurrence fails.

For any Harris recurrent set D, we write $D^\infty = \{y : L(y, D) = 1\}$, so that $D \subseteq D^\infty$, and D^∞ is absorbing.

We will call D a *maximal absorbing set* if $D = D^\infty$. This will be used, in general, in the following form:

Maximal Harris sets

We call a set H *maximal Harris* if H is a maximal absorbing set such that $\boldsymbol{\Phi}$ restricted to H is Harris recurrent.

Theorem 9.1.5 *If $\boldsymbol{\Phi}$ is recurrent, then we can write*

$$\mathsf{X} = H \cup N \tag{9.11}$$

where H is a non-empty maximal Harris set, and N is transient.

Proof Let C be a ψ_a-petite set in $\mathcal{B}^+(\mathsf{X})$, where we choose ψ_a as a maximal irreducibility measure. Set $H = \{y : Q(x, C) = 1\}$ and write $N = H^c$.

Clearly, since $H^\infty = H$, either H is empty or H is maximal absorbing. We first show that H is non-empty.

Suppose otherwise, so that $Q(x, C) < 1$ for all x. We first show this implies the set

$$C_1 := \{x \in C : L(x, C) < 1\} :$$

is in $\mathcal{B}^+(\mathsf{X})$.

For if not, and $\psi(C_1) = 0$, then by Proposition 4.2.3 there exists an absorbing full set $F \subset C_1^c$. We have by definition that $L(x, C) = 1$ for any $x \in C \cap F$, and since F is absorbing we must have $L(x, C \cap F) = 1$ for $x \in C \cap F$. From Proposition 9.1.1 it follows that $Q(x, C \cap F) = 1$ for $x \in C \cap F$, which gives a contradiction, since $Q(x, C) \geq Q(x, C \cap F)$. This shows that in fact $\psi(C_1) > 0$.

But now, since $C_1 \in \mathcal{B}^+(\mathsf{X})$ there exists $B \subseteq C_1, B \in \mathcal{B}^+(\mathsf{X})$ and $\delta > 0$ with $L(x, C_1) \leq \delta < 1$ for all $x \in B$: accordingly

$$L(x, B) \leq L(x, C_1) \leq \delta, \qquad x \in B.$$

Now Proposition 8.3.1 (iii) gives $U(x,B) \le [1-\delta]^{-1}, x \in B$ and this contradicts the assumed recurrence of $\boldsymbol{\Phi}$.

Thus H is a non-empty maximal absorbing set, and by Proposition 4.2.3 H is full: from Proposition 8.3.7 we have immediately that N is transient. It remains to prove that H is Harris.

For any set A in $\mathcal{B}^+(\mathsf{X})$ we have $C \rightsquigarrow A$. It follows from Theorem 9.1.3 that if $Q(x,C) = 1$ then $Q(x,A) = 1$ for every $A \in \mathcal{B}^+(\mathsf{X})$. Since by construction $Q(x,C) = 1$ for $x \in H$, we have also that $Q(x,A) = 1$ for any $x \in H$ and $A \in \mathcal{B}^+(\mathsf{X})$: so $\boldsymbol{\Phi}$ restricted to H is Harris recurrent, which is the required result. □

We now strengthen the connection between properties of $\boldsymbol{\Phi}$ and those of its skeletons.

Theorem 9.1.6 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible and aperiodic. Then $\boldsymbol{\Phi}$ is Harris if and only if each skeleton is Harris.*

Proof If the m-skeleton is Harris recurrent then, since $m\tau_A^m \ge \tau_A$ for any $A \in \mathcal{B}(\mathsf{X})$, where τ_A^m is the first entrance time for the m-skeleton, it immediately follows that $\boldsymbol{\Phi}$ is also Harris recurrent.

Suppose now that $\boldsymbol{\Phi}$ is Harris recurrent. For any $m \ge 2$ we know from Proposition 8.2.6 that $\boldsymbol{\Phi}^m$ is recurrent, and hence a Harris set H_m exists for this skeleton. Since H_m is full, there exists a subset $H \subset H_m$ which is absorbing and full for $\boldsymbol{\Phi}$, by Proposition 4.2.3.

Since $\boldsymbol{\Phi}$ is Harris recurrent we have that $\mathsf{P}_x\{\tau_H < \infty\} \equiv 1$, and since H is absorbing we know that $m\tau_H^m \le \tau_H + m$. This shows that

$$\mathsf{P}_x\{\tau_H^m < \infty\} = \mathsf{P}_x\{\tau_H < \infty\} \equiv 1$$

and hence $\boldsymbol{\Phi}^m$ is Harris recurrent as claimed. □

9.1.3 A hitting time criterion for Harris recurrence

The Harris recurrence results give useful extensions of the results in Theorem 8.3.5 and Theorem 8.3.6.

Proposition 9.1.7 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible.*

(i) *If some petite set C is recurrent, then $\boldsymbol{\Phi}$ is recurrent; and the set $C \cap N$ is uniformly transient, where N is the transient set in the Harris decomposition (9.11).*

(ii) *If there exists some petite set in $\mathcal{B}(\mathsf{X})$ such that $L(x,C) \equiv 1, x \in \mathsf{X}$, then $\boldsymbol{\Phi}$ is Harris recurrent.*

Proof **(i)** If C is recurrent then so is the chain, from Theorem 8.3.5. Let $D = C \cap N$ denote the part of C not in H. Since N is ψ-null, and ν is an irreducibility measure we must have $\nu(N) = 0$ by the maximality of ψ; hence (8.35) holds and from (8.37) we have a uniform bound on $U(x,D), x \in \mathsf{X}$ so that D is uniformly transient.

(ii) If $L(x,C) \equiv 1$, $x \in \mathsf{X}$ for some ψ_a-petite set C, then from Theorem 9.1.3 C is Harris recurrent. Since C is petite we have $C \rightsquigarrow A$ for each $A \in \mathcal{B}^+(\mathsf{X})$. The

Harris recurrence of C, together with Theorem 9.1.3 (ii), gives $Q(x,A) \equiv 1$ for all x, so $\boldsymbol{\Phi}$ is Harris recurrent. □

This leads to a stronger version of Theorem 8.4.3.

Theorem 9.1.8 *Suppose $\boldsymbol{\Phi}$ is a ψ-irreducible chain. If there exists a petite set $C \subset \mathsf{X}$, and a function V which is unbounded off petite sets such that (V1) holds then $\boldsymbol{\Phi}$ is Harris recurrent.*

PROOF In Theorem 8.4.3 we showed that $L(x, C \cup C_V(n)) \equiv 1$, for some n, so Harris recurrence has already been proved in view of Proposition 9.1.7. □

9.2 Non-evanescent and recurrent chains

9.2.1 Evanescence and transience

Let us now turn to chains on topological spaces. Here, as was the case when considering irreducibility, it is our major goal to delineate behavior on open sets rather than arbitrary sets in $\mathcal{B}(\mathsf{X})$; and when considering questions of stability in terms of sure return to sets, the objects of interest will typically be compact sets.

With probabilistic stability one has "finiteness" in terms of return visits to sets of positive measure of some sort, where the measure is often dependent on the chain; with topological stability the "finite" sets of interest are compact sets which are defined by the structure of the space rather than of the chain. It is obvious from the links between petite sets and compact sets for T-chains that we will be able to describe behavior on compacta directly from the behavior on petite sets described in the previous section, provided there is an appropriate continuous component for the transition law of $\boldsymbol{\Phi}$.

In this section we investigate a stability concept which provides such links between the chain and the topology on the space, and which we touched on in Section 1.3.1.

As we discussed in the introduction of this chapter, a sample path of $\boldsymbol{\Phi}$ is said to converge to infinity (denoted $\boldsymbol{\Phi} \to \infty$) if the trajectory visits each compact set only finitely often. Since X is locally compact and separable, it follows from Lindelöf's Theorem D.3.1 that there exists a countable collection of open precompact sets $\{O_n : n \in \mathbb{Z}_+\}$ such that

$$\{\boldsymbol{\Phi} \to \infty\} = \bigcap_{n=0}^{\infty} \{\boldsymbol{\Phi} \in O_n \ \text{ i.o.}\}^c.$$

In particular, then, the event $\{\boldsymbol{\Phi} \to \infty\}$ lies in $\mathcal{F}$.

Non-evanescent Chains

A Markov chain $\boldsymbol{\Phi}$ will be called *non-evanescent* if $\mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\} = 0$ for each $x \in \mathsf{X}$.

We first show that for a T-chain, either sample paths converge to infinity or they enter a recurrent part of the space. Recall that for any A, we have $A^0 = \{y : L(y, A) = 0\}$.

Theorem 9.2.1 *Suppose that $\boldsymbol{\Phi}$ is a T-chain. For any $A \in \mathcal{B}(\mathsf{X})$ which is transient, and for each $x \in \mathsf{X}$,*

$$\mathsf{P}_x\Big\{\{\boldsymbol{\Phi} \to \infty\} \cup \{\boldsymbol{\Phi} \text{ enters } A^0\}\Big\} = 1. \tag{9.12}$$

Thus if $\boldsymbol{\Phi}$ is a non-evanescent T-chain, then X is not transient.

PROOF Let $A = \bigcup B_j$, with each B_j uniformly transient; then from Proposition 8.3.2, the sets $\bar{B}_i(M) = \{x \in \mathsf{X} : \sum_{j=1}^{M} P^j(x, B_i) > M^{-1}\}$ are also uniformly transient, for any i, j. Thus $\bar{A} = \bigcup A_i$ where each A_i is uniformly transient.

Since T is lower semicontinuous, the sets $O_{ij} := \{x \in \mathsf{X} : T(x, A_i) > j^{-1}\}$ are open, as is $O_j := \{x \in \mathsf{X} : T(x, A^0) > j^{-1}\}$, $i, j \in \mathbb{Z}_+$. Since T is everywhere non-trivial we have for all $x \in \mathsf{X}$,

$$T(x, (\bigcup A_j) \cup A^0) = T(x, \mathsf{X}) > 0$$

and hence the sets $\{O_{ij}, O_j\}$ form an open cover of X.

Let C be a compact subset of X, and choose M such that $\{O_M, O_{iM} : 1 \le i \le M\}$ is a finite subcover of C. Since each A_i is uniformly transient, and

$$K_a(x, A_i) \ge T(x, A_i) \ge j^{-1} \qquad x \in O_{ij} \tag{9.13}$$

we know from Proposition 8.3.2 that each of the sets O_{ij} is uniformly transient. It follows that with probability one, every trajectory that enters C infinitely often must enter O_M infinitely often: that is,

$$\{\boldsymbol{\Phi} \in C \text{ i.o.}\} \subset \{\boldsymbol{\Phi} \in O_M \text{ i.o.}\} \qquad \text{a.s.} \quad [\mathsf{P}_*]$$

But since $L(x, A^0) > 1/M$ for $x \in O_M$ we have by Theorem 9.1.3 that

$$\{\boldsymbol{\Phi} \in O_M \text{ i.o.}\} \subset \{\boldsymbol{\Phi} \in A^0 \text{ i.o.}\} \qquad \text{a.s.} \quad [\mathsf{P}_*]$$

and this completes the proof of (9.12). □

9.2.2 Non-evanescence and recurrence

We can now prove one of the major links between topological and probabilistic stability conditions.

Theorem 9.2.2 *For a ψ-irreducible T-chain, the space admits a decomposition*

$$\mathsf{X} = H \cup N$$

where H is either empty or a maximal Harris set, and N is transient: and for all $x \in \mathsf{X}$,

$$L(x, H) = 1 - \mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\}. \tag{9.14}$$

Hence we have

(i) *the chain is recurrent if and only if $\mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\} < 1$ for some $x \in \mathsf{X}$; and*

(ii) *the chain is Harris recurrent if and only if the chain is non-evanescent.*

PROOF We have the decomposition $\mathsf{X} = H \cup N$ from Theorem 9.1.5 in the recurrent case, and Theorem 8.3.4 otherwise.

We have (9.14) from (9.12), since N is transient and $H = N^0$.

Thus if $\boldsymbol{\Phi}$ is a non-evanescent T-chain, then it must leave the transient set N in (9.11) with probability one, from Theorem 9.2.1. By construction, this means N is empty, and $\boldsymbol{\Phi}$ is Harris recurrent.

Conversely, if $\boldsymbol{\Phi}$ is Harris recurrent (9.14) shows the chain is non-evanescent. □

This result shows that natural definitions of stability and instability in the topological and in the probabilistic contexts are exactly equivalent, for chains appropriately adapted to the topology.

Before exploring conditions for either recurrence or non-evanescence, we look at the ways in which it is possible to classify individual states on a topological space, and the solidarity between such definitions and the overall classification of the chain which we have just described.

9.3 Topologically recurrent and transient states

9.3.1 Classifying states through neighborhoods

We now introduce some natural stochastic stability concepts for individual states when the space admits a topology. The reader should be aware that uses of terms such as "recurrence" vary across the literature. Our definitions are consistent with those we have given earlier, and indeed will be shown to be identical under appropriate conditions when the chain is an irreducible T-chain or an irreducible Feller process; however, when comparing them with some terms used by other authors, care needs to be taken.

In the general space case, we developed definitions for sets rather than individual states: when there is a topology, and hence a natural collection of sets (the open neighborhoods) associated with each point, it is possible to discuss recurrence and transience of each point even if each point is not itself reached with positive probability.

Topological recurrence concepts

We shall call a point x^* *topologically recurrent* if $U(x^*, O) = \infty$ for all neighborhoods O of x^*, and *topologically transient* otherwise.

We shall call a point x^* *topologically Harris recurrent* if $Q(x^*, O) = 1$ for all neighborhoods O of x^*.

We first determine that this definition of topological Harris recurrence is equivalent to the formally weaker version involving finiteness only of first return times.

Proposition 9.3.1 *The point x^* is topologically Harris recurrent if and only if $L(x^*, O) = 1$ for all neighborhoods O of x^*.*

PROOF Our assumption is that

$$\mathsf{P}_{x^*}(\tau_O < \infty) = 1, \tag{9.15}$$

for each neighborhood O of x^*. We show by induction that if $\tau_O(j)$ is the time of the j^{th} return to O as usual, and for some integer $j \geq 1$,

$$\mathsf{P}_{x^*}(\tau_O(j) < \infty) = 1, \tag{9.16}$$

for each neighborhood O of x^*, then for each such neighborhood

$$\mathsf{P}_{x^*}(\tau_O(j+1) < \infty) = 1. \tag{9.17}$$

Thus (9.17) holds for all j and the point x^* is by definition topologically Harris recurrent.

Recall that for any $B \subset O$ we have the following probabilistic interpretation of the kernel U_O:

$$U_O(x^*, B) = \mathsf{P}_{x^*}(\tau_O < \infty \quad \text{and} \quad \Phi_{\tau_O} \in B)$$

Suppose that $U_O(x^*, \{x^*\}) = q \geq 0$ where $\{x^*\}$ is the set containing the one point x^*, so that

$$U_O(x^*, O \backslash \{x^*\}) = 1 - q. \tag{9.18}$$

The assumption that j distinct returns to O are sure implies that

$$\mathsf{P}_{x^*}(\Phi_{\tau_O(1)} = x^*, \Phi_{\tau_O(r)} \in O, r = 2, \ldots, j+1) = q. \tag{9.19}$$

Let $O_d \downarrow \{x^*\}$ be a countable neighborhood basis at x^*. The assumption (9.16) applied to each O_d also implies that

$$\mathsf{P}_y(\tau_{O_d}(j) < \infty) = 1, \tag{9.20}$$

for almost all y in $O \backslash O_d$ with respect to $U_O(x^*, \cdot)$. But by (9.18) we have

$$U_O(x^*, O \backslash O_d) \uparrow 1 - q,$$

as $O_d \downarrow \{x^*\}$ and so by (9.20),

$$\begin{array}{rcl} \int_{O \backslash \{x^*\}} U_O(x, dy) \mathsf{P}_y(\tau_O(j) < \infty) & \geq & \lim_{d \downarrow 0} \int_{O \backslash O_d} U_O(x^*, dy) \mathsf{P}_y(\tau_{O_d}(j) < \infty) \\ & = & 1 - q. \end{array} \tag{9.21}$$

This yields the desired conclusion, since by (9.19) and (9.21),

$$\mathsf{P}_{x^*}(\tau_O(j+1) < \infty) = \int_O U_O(x^*, dy) \mathsf{P}_y(\tau_O(j) < \infty) = 1.$$

□

9.3.2 Solidarity of recurrence for T-chains

For T-chains we can connect the idea of properties of individual states with the properties of the whole space under suitable topological irreducibility conditions.

The key to much of our analysis of chains on topological spaces is the following simple lemma.

Lemma 9.3.2 *If $\boldsymbol{\Phi}$ is a T-chain, and $T(x^*, B) > 0$ for some x^*, B, then there is a neighborhood O of x^* and a distribution a such that $O \overset{a}{\rightsquigarrow} B$, and hence from Lemma 5.5.1, $O \rightsquigarrow B$.*

PROOF Since $\boldsymbol{\Phi}$ is a T-chain, there exists some distribution a such that for all x,

$$K_a(x, B) \geq T(x, B).$$

But since $T(x^*, B) > 0$ and $T(x, B)$ is lower semicontinuous, it follows that for some neighborhood O of x^*,

$$\inf_{x \in O} T(x, B) > 0$$

and thus, as in (5.45),

$$\inf_{x \in O} L(x, B) \geq \inf_{x \in O} K_a(x, B) \geq \inf_{x \in O} T(x, B)$$

and the result is proved. $\square$

Theorem 9.3.3 *Suppose that $\boldsymbol{\Phi}$ is a ψ-irreducible T-chain, and that x^* is reachable. Then $\boldsymbol{\Phi}$ is recurrent if and only if x^* is topologically recurrent.*

PROOF If x^* is reachable then $x^* \in \operatorname{supp} \psi$ and so $O \in \mathcal{B}^+(\mathsf{X})$ for every neighborhood of x^*. Thus if $\boldsymbol{\Phi}$ is recurrent then every neighborhood O of x^* is recurrent, and so by definition x^* is topologically recurrent.

If $\boldsymbol{\Phi}$ is transient then there exists a uniformly transient set B such that $T(x^*, B) > 0$, from Theorem 8.3.4, and thus from Lemma 9.3.2 there is a neighborhood O of x^* such that $O \rightsquigarrow B$; and now from Proposition 8.3.2, O is uniformly transient and thus x^* is topologically transient also. $\square$

We now work towards developing links between topological recurrence and topological Harris recurrence of points, as we did with sets in the general space case.

It is unfortunately easy to construct an example which shows that even for a T-chain, topologically recurrent states need not be topologically Harris recurrent without some extra assumptions. For take $\mathsf{X} = [0, 1] \cup \{2\}$, and define the transition law for $\boldsymbol{\Phi}$ by

$$\begin{array}{rcll} P(0, \cdot\,) & = & (\mu + \delta_2)/2 & \\ P(x, \cdot\,) & = & \mu, & x \in (0, 1] \\ P(2, \cdot\,) & = & \delta_2 & \end{array} \tag{9.22}$$

where μ is Lebesgue measure on $[0, 1]$ and δ_2 is the point mass at $\{2\}$. Set the everywhere non-trivial continuous component T of P itself as

$$\begin{array}{rcll} T(x, \cdot\,) & = & \mu/2, & x \in [0, 1] \\ T(2, \cdot\,) & = & \delta_2. & \end{array} \tag{9.23}$$

By direct calculation one can easily see that $\{0\}$ is a topologically recurrent state but is not topologically Harris recurrent.

It is also possible to develop examples where the chain is weak Feller but topological recurrence does not imply topological Harris recurrence of states.

Let $\mathsf{X} = \{0, \pm 1, \pm 2, \ldots, \pm\infty\}$, and choose $0 < p < \frac{1}{2}$ and $q = 1 - p$. Put $P(0,1) = p, P(0,-1) = q$, and for $n = 1, 2, \ldots$ set

$$\begin{array}{rclrclrcl} P(n,n+1) & = & p & P(n,n-1) & = & q & & & \\ P(-n,-n-1) & = & p & P(-n,0) & = & \frac{1}{2}-p & P(-n,n) & = & \frac{1}{2} \\ P(-\infty,-\infty) & = & p & P(-\infty,0) & = & \frac{1}{2}-p & P(-\infty,\infty) & = & \frac{1}{2} \\ P(\infty,\infty) & = & 1. & & & & & & \end{array} \tag{9.24}$$

By comparison with a simple random walk, such as that analyzed in Proposition 8.4.4, it is clear that the finite integers are all recurrent states in the countable state space sense.

Now endow the space X with the discrete topology on the integers, and with a countable basis for the neighborhoods at $\infty, -\infty$ given respectively by the sets $\{n, n+1, \ldots, \infty\}$ and $\{-n, -n-1, \ldots, -\infty\}$ for $n \in \mathbb{Z}_+$. The chain is a Feller chain in this topology, and every neighborhood of $-\infty$ is recurrent so that $-\infty$ is a topologically recurrent state.

But $L(-\infty, \{-\infty, -1\}) < \frac{1}{2}$, so the state at $-\infty$ is not topologically Harris recurrent.

There are however some connections which do hold between recurrence and Harris recurrence.

Proposition 9.3.4 *If Φ is a T-chain and the state x^* is topologically recurrent then $Q(x^*, O) > 0$ for all neighborhoods O of x^*.*

If $P(x^*, \cdot\,) \cong T(x^*, \cdot\,)$ then also x^* is topologically Harris recurrent. In particular, therefore, for strong Feller chains topologically recurrent states are topologically Harris recurrent.

PROOF **(i)** Assume the state x^* is topologically recurrent but that O is a neighborhood of x^* with $Q(x^*, O) = 0$. Let $O^\infty = \{y : Q(y, O) = 1\}$, so that $L(x^*, O^\infty) = 0$. Since

$$L(x, A) \geq K_a(x, A) \geq T(x, A), \qquad x \in \mathsf{X}, \quad A \in \mathcal{B}(\mathsf{X})$$

this implies $T(x^*, O^\infty) = 0$, and since T is non-trivial, we must have

$$T(x^*, [O^\infty]^c) > 0. \tag{9.25}$$

Let $D_n := \{y : \mathsf{P}_y(\eta_O < n) > n^{-1}\}$: since $D_n \uparrow [O^\infty]^c$, we must have $T(x^*, D_n) > 0$ for some n. The continuity of T now ensures that there exists some δ and a neighborhood $O_\delta \subseteq O$ of x^* such that

$$T(x, D_n) > \delta, \qquad x \in O_\delta. \tag{9.26}$$

Let us take m large enough that $\sum_m^\infty a(j) \leq \delta/2$: then from (9.26) we have

$$\max_{1 \leq j \leq m} P^j(x, D_n) > \delta/2m, \qquad x \in O_\delta, \tag{9.27}$$

which obviously implies

$$\mathsf{P}_x(\tau_{D_n} \leq m) > \delta/2m, \qquad x \in O_\delta. \tag{9.28}$$

It follows that

$$\begin{aligned} \mathsf{P}_x(\eta_{O_\delta} \leq m+n) &\geq \mathsf{P}_x(\eta_O \leq m+n) \\ &\geq \textstyle\sum_1^m \int_{D_n} {}_{D_n}P^k(x,dy)\mathsf{P}_y(\eta_O \leq n) \\ &\geq n^{-1}\mathsf{P}(\tau_{D_n} \leq m) \\ &\geq n^{-1}\delta/2m, \qquad x \in O_\delta. \end{aligned} \tag{9.29}$$

With (9.29) established we can apply Proposition 8.3.1 to see that O_δ is uniformly transient.

This contradicts our assumption that x^* is topologically recurrent, and so in fact $Q(x^*, O) > 0$ for all neighborhoods O.

(ii) Suppose now that $P(x^*, \cdot\,)$ and $T(x^*, \cdot\,)$ are equivalent. Choose x^* topologically recurrent and assume we can find a neighborhood O with $Q(x^*, O) < 1$. Define O^∞ as before, and note that now $P(x^*, [O^\infty]^c) > 0$ since otherwise

$$Q(x^*, O) \geq \int_{O^\infty} P(x^*, dy)Q(y, O) = 1;$$

and so also $T(x^*, [O^\infty]^c) > 0$. Thus we again have (9.25) holding, and the argument in (i) shows that there is a uniformly transient neighborhood of x^*, again contradicting the assumption of topological recurrence. Hence x^* is topologically Harris recurrent.

□

The examples (9.22) and (9.24) show that we do not get, in general, the second conclusion of this proposition if the chain is merely weak Feller or has only a strong Feller component.

In these examples, it is the lack of irreducibility which allows such obvious "pathological" behavior, and we shall see in Theorem 9.3.6 that when the chain is a ψ-irreducible T-chain then this behavior is excluded. Even so, without any irreducibility assumptions we are able to derive a reasonable analogue of Theorem 9.1.5, showing that the non-Harris recurrent states form a transient set.

Theorem 9.3.5 *For any chain $\boldsymbol{\Phi}$ there is a decomposition*

$$\mathsf{X} = R \cup N$$

where R denotes the set of states which are topologically Harris recurrent, and N is transient.

PROOF Let O_i be a countable basis for the topology on X. If $x \in R^c$ then, by Proposition 9.3.1, we have some $n \in \mathbb{Z}_+$ such that $x \in O_n$ with $L(x, O_n) < 1$. Thus the sets $D_n = \{y \in O_n : L(y, O_n) < 1\}$ cover the set of non-topologically Harris recurrent states. We can further partition each D_n into

$$D_n(j) := \{y \in D_n : L(y, O_n) \leq 1 - j^{-1}\}$$

and by this construction, for $y \in D_n(j)$ we have

$$L(y, D_n(j)) \leq L(y, D_n) \leq L(y, O_n) \leq 1 - j^{-1} :$$

it follows from Proposition 8.3.1 that $U(x, D_n(j))$ is bounded above by j, and hence is uniformly transient. □

Regrettably, this decomposition does not partition X into Harris recurrent and transient states, since the sets $D_n(j)$ in the cover of non-Harris states may not be open. Therefore there may actually be topologically recurrent states which lie in the set which we would hope to have as the "transient" part of the space, as happens in the example (9.22).

We can, for ψ-irreducible T-chains, now improve on this result to round out the links between the Harris properties of points and those of the chain itself.

Theorem 9.3.6 *For a ψ-irreducible T-chain, the space admits a decomposition*

$$\mathsf{X} = H \cup N$$

where H is non-empty or a maximal Harris set and N is transient; the set of Harris recurrent states R is contained in H; and every state in N is topologically transient.

Proof The decomposition has already been shown to exist in Theorem 9.2.2. Let $x^* \in R$ be a topologically Harris recurrent state. Then from (9.14), we must have $L(x, H) = 1$, and so $x^* \in H$ by maximality of H.

We can write $N = N_E \cup N_H$ where $N_H = \{y \in N : T(y, H) > 0\}$ and $N_E = \{y \in N : T(y, H) = 0\}$. For fixed $x^* \in N_H$ there exists $\delta > 0$ and an open set O_δ such that $x^* \in O_\delta$ and $T(y, H) > \delta$ for all $y \in O_\delta$, by the lower semicontinuity of $T(\,\cdot\,, H)$.

Hence also the sampled kernel K_a minorized by T satisfies $K_a(y, H) > \delta$ for all $y \in O_\delta$. Now choose M such that $\sum_{n>M} a(n) \leq \delta/2$. Then for all $y \in O_\delta$

$$\sum_{n \leq M} P^n(y, H) a(n) \geq \delta/2$$

and since H is absorbing

$$\mathsf{P}_y(\eta_N > M) = \mathsf{P}_y(\tau_H > M) \leq 1 - \delta/2$$

which shows that O_δ is uniformly transient from (8.37).

If on the other hand $x^* \in N_E$ then since T is non-trivial, there exists a uniformly transient set $D \subseteq N$ such $T(x^*, D) > 0$; and now by Lemma 9.3.2, there is again a neighbourhood O of x^* with $O \overset{a}{\rightsquigarrow} D$, so that O is uniformly transient by Proposition 8.3.2 as required. □

The maximal Harris set in Theorem 9.3.6 may be strictly larger than the set R of topologically Harris recurrent states. For consider the trivial example where $\mathsf{X} = [0, 1]$ and $P(x, \{0\}) = 1$ for all x. This is a δ_0-irreducible strongly Feller chain, with $R = \{0\}$ and yet $H = [0, 1]$.

9.4 Criteria for stability on a topological space

9.4.1 A drift criterion for non-evanescence

We can extend the results of Theorem 8.4.3 in a number of ways if we take up the obvious martingale implications of (V1), and in the topological case we can also gain a better understanding of the rather inexplicit concept of functions unbounded off petite sets for a particular chain if we define "norm-like" functions.

Norm-like Functions

A function V is called *norm-like* if $V(x) \to \infty$ as $x \to \infty$: this means that the sublevel sets $\{x : V(x) \le r\}$ are precompact for each $r > 0$.

This nomenclature is designed to remind the user that we seek functions which behave like norms: they are large as the distance from the center of the space increases. Typically in practice, a norm-like function will be a norm on Euclidean space, or at least a monotone function of a norm. For irreducible T-chains, functions unbounded off petite sets certainly include norm-like functions, since compacta are petite in that case; but of course norm-like functions are independent of the structure of the chain itself.

Even without irreducibility we get a useful conclusion from applying (V1).

Theorem 9.4.1 *If condition (V1) holds for a norm-like function V and a compact set C then $\boldsymbol{\Phi}$ is non-evanescent.*

PROOF Suppose that in fact $\mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\} > 0$ for some $x \in \mathsf{X}$. Then, since the set C is compact, there exists $M \in \mathbb{Z}_+$ with

$$\mathsf{P}_x\{\{\Phi_k \in C^c, k \ge M\} \cap \{\boldsymbol{\Phi} \to \infty\}\} > 0.$$

Hence letting $\mu = P^M(x, \cdot)$, we have by conditioning at time M,

$$\mathsf{P}_\mu\{\{\sigma_C = \infty\} \cap \{\boldsymbol{\Phi} \to \infty\}\} > 0. \tag{9.30}$$

We now show that (9.30) leads to a contradiction.

In order to use the martingale nature of (V1), we write (8.44) as

$$\mathsf{E}[V(\Phi_{k+1}) \mid \mathcal{F}_k^{\Phi}] \le V(\Phi_k) \qquad \text{a.s. } [\mathsf{P}_*],$$

when $\sigma_C > k$, $k \in \mathbb{Z}_+$.

Now let $M_i = V(\Phi_i)\mathbb{1}\{\sigma_C \ge i\}$. Using the fact that $\{\sigma_C \ge k\} \in \mathcal{F}_{k-1}^{\Phi}$, we may show that $(M_k, \mathcal{F}_k^{\Phi})$ is a positive supermartingale: indeed,

$$\mathsf{E}[M_k \mid \mathcal{F}_{k-1}^{\Phi}] = \mathbb{1}\{\sigma_C \ge k\}\mathsf{E}[V(\Phi_k) \mid \mathcal{F}_{k-1}^{\Phi}] \le \mathbb{1}\{\sigma_C \ge k\}V(\Phi_{k-1}) \le M_{k-1}.$$

Hence there exists an almost surely finite random variable M_∞ such that $M_k \to M_\infty$ as $k \to \infty$.

There are two possibilities for the limit M_∞. Either $\sigma_C < \infty$ in which case $M_\infty = 0$, or $\sigma_C = \infty$ in which case $\limsup_{k\to\infty} V(\Phi_k) = M_\infty < \infty$ and in particular $\boldsymbol{\Phi} \not\to \infty$ since V is norm-like. Thus we have shown that

$$\mathsf{P}_\mu\{\{\sigma_C < \infty\} \cup \{\boldsymbol{\Phi} \to \infty\}^c\} = 1,$$

which clearly contradicts (9.30). Hence $\boldsymbol{\Phi}$ is non-evanescent. □

Note that in general the set C used in (V1) is not necessarily Harris recurrent, and it is possible that the set may not be reached from any initial condition. Consider the example where $\mathsf{X} = \mathbb{R}_+$, $P(0, \{1\}) = 1$, and $P(x, \{x\}) \equiv 1$ for $x > 0$. This is non-evanescent, satisfies (V1) with $V(x) = x$, and $C = \{0\}$, but clearly from x there is no possibility of reaching compacta not containing $\{x\}$.

However, from our previous analysis in Theorem 9.1.8 we obviously have that if $\boldsymbol{\Phi}$ is ψ-irreducible and Condition (V1) holds for C petite, then both C and $\boldsymbol{\Phi}$ are Harris recurrent.

9.4.2 A converse theorem for Feller chains

In the topological case we can construct a converse to the drift condition (V1), provided the chain has appropriate continuity properties.

Theorem 9.4.2 *Suppose that $\boldsymbol{\Phi}$ is a weak Feller chain, and suppose that there exists a compact set C satisfying $\sigma_C < \infty$ a.s. $[\mathsf{P}_*]$.*

Then there exists a compact set C_0 containing C and a norm-like function V, bounded on compacta, such that

$$\Delta V(x) \le 0, \qquad x \in C_0^c. \tag{9.31}$$

PROOF Let $\{A_n\}$ be a countable increasing cover of X by open precompact sets with $C \subseteq A_0$; and put $D_n = A_n^c$ for $n \in \mathbb{Z}_+$. For $n \in \mathbb{Z}_+$, set

$$V_n(x) = \mathsf{P}_x(\sigma_{D_n} < \sigma_{A_0}). \tag{9.32}$$

For any fixed n and any $x \in A_0^c$ we have from the Markov property that the sequence $V_n(x)$ satisfies, for $x \in A_0^c \cap D_n^c$

$$\begin{aligned} \textstyle\int P(x,dy)V_n(y) &= \mathsf{E}_x[\mathsf{P}_{\Phi_1}\{\sigma_{D_n} < \sigma_{A_0}\}] \\ &= \mathsf{P}_x\{\sigma_{D_n} < \sigma_{A_0}\} \\ &= V_n(x) \end{aligned} \tag{9.33}$$

whilst for $x \in D_n$ we have $V_n(x) = 1$; so that for all $n \in \mathbb{Z}_+$ and $x \in A_0^c$

$$\int P(x,dy)V_n(y) \le V_n(x). \tag{9.34}$$

We will show that for suitably chosen $\{n_i\}$ the function

$$V(x) = \sum_{i=0}^{\infty} V_{n_i}(x), \tag{9.35}$$

which clearly satisfies the appropriate drift condition by linearity from (9.34) if finitely defined, gives the required converse result.

Since $V_n(x) = 1$ on D_n, it is clear that V is norm-like. To complete the proof we must show that the sequence $\{n_i\}$ can be chosen to ensure that V is bounded on compact sets, and it is for this we require the Feller property.

Let $m \in \mathbb{Z}_+$ and take the upper bound

$$\begin{aligned} V_n(x) &= \mathsf{P}_x\{\{\sigma_{D_n} < \sigma_{A_0}\} \cap \{\sigma_{A_0} \le m\} \cup \{\sigma_{D_n} < \sigma_{A_0}\} \cap \{\sigma_{A_0} > m\}\} \\ &\le \mathsf{P}_x\{\sigma_{D_n} < m\} + \mathsf{P}_x\{\sigma_{A_0} > m\}. \end{aligned} \tag{9.36}$$

Choose the sequence $\{n_i\}$ as follows. By Proposition 6.1.1, the function $\mathsf{P}_x\{\sigma_{A_0} > m\}$ is an upper semi-continuous function of x, which converges to zero as $m \to \infty$ for all x. Hence the convergence is uniform on compacta, and thus we can choose m_i so large that

$$\mathsf{P}_x\{\sigma_{A_0} > m_i\} < 2^{-(i+1)}, \qquad x \in A_i. \tag{9.37}$$

Now for m_i fixed for each i, consider $\mathsf{P}_x\{\sigma_{D_n} < m_i\}$: as a function of x this is also upper semi-continuous and converges to zero as $n \to \infty$ for all x. Hence again we see that the convergence is uniform on compacta, which implies we may choose n_i so large that

$$\mathsf{P}_x\{\sigma_{D_{n_i}} < m_i\} < 2^{-(i+1)}, \qquad x \in A_i. \tag{9.38}$$

Combining (9.36), (9.37) and (9.38) we see that $V_{n_i} \le 2^{-i}$ for $x \in A_i$. From (9.35) this implies, finally, for all $k \in \mathbb{Z}_+$ and $x \in A_k$

$$\begin{aligned} V(x) &\le k + \sum_{i=k}^{\infty} V_{n_i}(x) \\ &\le k + \sum_{i=k}^{\infty} 2^{-i} \\ &\le k + 1 \end{aligned} \tag{9.39}$$

which completes the proof. □

The following somewhat pathological example shows that in this instance we cannot use a strongly continuous component condition in place of the Feller property if we require V to be continuous.

Set $\mathsf{X} = \mathbb{R}_+$ and for every irrational x and every integer x set $P(x, \{0\}) = 1$. Let $\{r_n\}$ be an ordering of the remaining rationals $\mathbb{Q}\backslash\mathbb{Z}_+$, and define P for these states by $P(r_n, 0) = 1/2$, $P(r_n, n) = 1/2$. Then the chain is δ_0-irreducible, and clearly recurrent; and the component $T(x, A) = \frac{1}{2}\delta_0\{A\}$ renders the chain a T-chain. But $PV(r_n) \ge V(n)/2$, so that for any norm-like function V, within any open set $\int P(x, dy)V(y)$ is unbounded.

However, for discontinuous V we do get a norm-like test function: just take $V(r_n) = n$, and $V(x) = x$, for x not equal to any r_n. Then $PV(r_n) = n/2 < V(r_n)$, and $PV(x) = 0 < V(x)$, for x not equal to any r_n, so that (V1) does hold.

9.4.3 Non-evanescence of random walk

As an example of the use of (V1) we consider in more detail the analysis of the unrestricted random walk

$$\Phi_n = \Phi_{n-1} + W_n.$$

We will show that if W is an increment variable on $\mathbb{R}$ with $\beta = 0$ and

$$\mathsf{E}(W^2) = \int w^2\, \Gamma(dw) < \infty$$

then the unrestricted random walk on $\mathbb{R}$ with increment W is non-evanescent.

To verify this using (V1) we first need to add to the bounds on the moments of Γ which we gave in Lemma 8.5.2 and Lemma 8.5.3.

Lemma 9.4.3 *Let W be a random variable, s a positive number and t any real number. Then for any $B \subseteq \{w : -s + tw > 0\}$,*

$$\mathsf{E}[\log(-s + tW)\mathbb{1}\{W \in B\}] \leq \mathsf{P}(B)(\log(s) - 2) + (t/s)\mathsf{E}[W\mathbb{1}\{W \in B\}].$$

PROOF For all $x > 1$, $\log(-1 + x) \leq x - 2$. Thus

$$\begin{aligned} \log(-s + tW)\mathbb{1}\{W \in B\} &= [\log(s) + \log(-1 + tW/s)]\mathbb{1}\{W \in B\} \\ &\leq (\log(s) + tW/s - 2)\mathbb{1}\{W \in B\}; \end{aligned}$$

taking expectations again gives the result. □

Lemma 9.4.4 *Let W be a random variable with distribution function Γ and finite variance. Let s, c, u_2, and v_2 be positive numbers, and let $t_1 \geq t_2$ and u_1, v_1, t be real numbers. Then*

(i)

$$\lim_{x \to -\infty} x^2[-\Gamma(-\infty, t_1 + sx)\log(u_1 - u_2 x) + \Gamma(-\infty, t_2 + sx)(\log(v_1 - v_2 x) - c)] \leq 0. \tag{9.40}$$

(ii)

$$\lim_{x \to \infty} x^2[-\Gamma(t_2 + sx, \infty)\log(v_1 + v_2 x) + \Gamma(t_1 + sx, \infty)(\log(u_1 + u_2 x) - c)] \leq 0. \tag{9.41}$$

PROOF To see (i), note that from

$$\lim_{x \to \infty} x^2 \Gamma(-\infty, t_2 + sx) = 0$$

and

$$\lim_{x \to \infty} \log[(u_1 - u_2 x)/(v_1 - v_2 x)] = \log(u_2/v_2),$$

we have

$$\begin{aligned} &\lim_{x \to \infty} x^2 \Big[-\Gamma(-\infty, t_1 + sx)\log(u_1 - u_2 x) + \Gamma(-\infty, t_2 + sx)(\log(v_1 - v_2 x) - c)\Big] \\ &= \lim_{x \to \infty} \Big[-x^2(\Gamma(-\infty, t_1 + sx) - \Gamma(-\infty, t_2 + sx))\log(u_1 - u_2 x)\Big] \\ &\qquad \Big[-x^2 \Gamma(-\infty, t_2 + sx)\log[(u_1 - u_2 x)/(v_1 - v_2 x)] - c x^2 \Gamma(-\infty, t_2 + sx)\Big] \end{aligned}$$

which is non-positive. The proof of (ii) is similar. □

We can now prove the most general version of Theorem 8.1.5 using a drift condition that we shall attempt.

Proposition 9.4.5 *If W is an increment variable on $\mathbb{R}$ with $\beta = 0$ and $\mathsf{E}(W^2) < \infty$ then the unrestricted random walk on $\mathbb{R}_+$ with increment W is non-evanescent.*

PROOF In this situation we use the test function

$$V(x) = \begin{cases} \log(1+x) & x > R \\ \log(1-x) & x < -R \end{cases} \tag{9.42}$$

and $V(x) = 0$ in the region $[-R, R]$, where $R > 1$ is again a positive constant to be chosen.

We need to evaluate the behavior of $\mathsf{E}_x[V(X_1)]$ near both ∞ and $-\infty$ in this case, and we write

$$\begin{aligned} V_1(x) &= \mathsf{E}_x[\log(1+x+W)\mathbb{1}\{x+W>R\}] \\ V_2(x) &= \mathsf{E}_x[\log(1-x-W)\mathbb{1}\{x+W<-R\}] \end{aligned} \tag{9.43}$$

so that

$$\mathsf{E}_x[V(X_1)] = V_1(x) + V_2(x).$$

This time we develop bounds using the functions

$$\begin{aligned} V_3(x) &= (1/(1+x))\mathsf{E}[W\mathbb{1}\{W>R-x\}] \\ V_4(x) &= (1/(2(1+x)^2))\mathsf{E}[W^2\mathbb{1}\{R-x<W<0\}] \\ V_5(x) &= (1/(1-x))\mathsf{E}[W\mathbb{1}\{W<-R-x\}]. \end{aligned} \tag{9.44}$$

For $x > R, 1+x > 0$, and thus as in (8.61), by Lemma 8.5.2,

$$V_1(x) \leq \Gamma(R-x,\infty)\log(1+x) + V_3(x) - V_4(x),$$

while $1-x<0$, and by Lemma 9.4.3,

$$V_2(x) \leq \Gamma(-\infty,-R-x)(\log(-1+x)-2) - V_5(x).$$

Since $\mathsf{E}(W^2) < \infty$

$$V_4(x) = (1/(2(1+x)^2))\mathsf{E}[W^2\mathbb{1}\{W<0\}] - o(x^{-2}),$$

and by Lemma 8.5.3, both V_3 and V_5 are also $o(x^{-2})$. By Lemma 9.4.4 (i) we also have

$$-\Gamma(-\infty,R-x)\log(1+x) + \Gamma(-\infty,-R-x)(\log(-1+x)-2) \leq o(x^{-2}).$$

Thus by choosing R large enough

$$\begin{aligned} \mathsf{E}_x[V(X_1)] &\leq V(x) - (1/(2(1+x)^2))\mathsf{E}[W^2\mathbb{1}\{W<0\}] + o(x^{-2}) \\ &\leq V(x), \qquad x > R. \end{aligned} \tag{9.45}$$

The situation with $x < -R$ is exactly symmetric, and thus we have that V is a norm-like function satisfying (V1); and so the chain is non-evanescent from Theorem 9.4.1. □

9.5 Stochastic comparison and increment analysis

There are two further valuable tools for analyzing specific chains which we will consider in this final section on recurrence and transience. Both have been used implicitly in some of the examples we have looked at in this and the previous chapter, but because they are of wide applicability we will discuss them somewhat more formally here.

The first method analyzes chains through an "increment analysis". Because they consider only expected changes in the one-step position of some function V of the chain, and because expectation is a linear operator, drift criteria such as those in Section 9.4 essentially classify the behavior of the Markov model by a linearization of its increments. They are therefore often relatively easy to use for models where the transitions are already somewhat linear in structure, such as those based on the random walk: we have already seen this in our analysis of random walk on the half line in Section 8.4.3.

Such increment analysis is of value in many models, especially if combined with "stochastic comparison" arguments, which rely heavily on the classification of chains through return time probabilities.

In this section we will further use the stochastic comparison approach to discuss the structure of scalar linear models and general random walk on $\mathbb{R}$, and the special nonlinear SETAR models; we will then consider an increment analysis of general models on $\mathbb{R}_+$ which have no inherent linearity in their structure.

9.5.1 Linear models and the stochastic comparison technique

Suppose we have two φ-irreducible chains $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}'$ evolving on a common state space, and that for some set C and for all n

$$\mathsf{P}_x(\tau_C \geq n) \leq \mathsf{P}'_x(\tau_C \geq n), \qquad x \in C^c. \tag{9.46}$$

This is not uncommon if the chains have similarly defined structure, as is the case with random walk and the associated walk on a half line.

The stochastic comparison method tells us that a classification of one of the chains may automatically classify the other.

In one direction we have, provided C is a petite set for both chains, that when $\mathsf{P}'_x(\tau_C \geq n) \to 0$ as $n \to \infty$ for $x \in C^c$, then not only is $\boldsymbol{\Phi}'$ Harris recurrent, but $\boldsymbol{\Phi}$ is also Harris recurrent.

This is obvious. Its value arises in cases where the first chain $\boldsymbol{\Phi}'$ has a (relatively) simpler structure so that its analysis is straightforward through, say, drift conditions, and when the validation of (9.46) is also relatively easy.

In many ways stochastic comparison arguments are even more valuable in the transient context: as we have seen with random walk, establishing transience may need a rather delicate argument, and it is then useful to be able to classify "more transient" chains easily.

Suppose that (9.46) holds, and again that C is a φ-irreducible petite set for both chains. Then if $\boldsymbol{\Phi}$ is transient, we know that from Theorem 8.3.6 that there exists $D \subset C^c$ such that $L(x, C) < 1 - \varepsilon$ for $x \in D$ where $\varphi(D) > 0$; it then follows that $\boldsymbol{\Phi}'$ is also transient.

We first illustrate the strengths and drawbacks of this method in proving transience for the general random walk on the half line $\mathbb{R}_+$.

Proposition 9.5.1 *If Φ is random walk on $\mathbb{R}_+$ and if $\beta > 0$ then Φ is transient.*

PROOF Consider the discretized version W_h of the increment variable W with distribution

$$\mathsf{P}(W_h = nh) = \Gamma_h(nh)$$

where $\Gamma_h(nh)$ is constructed by setting, for every n

$$\Gamma_h(nh) = \int_{nh}^{(n+1)h} \Gamma(dw),$$

and let Φ_h be the corresponding random walk on the countable half line $\{nh, n \in \mathbb{Z}_+\}$. Then we have firstly that for any starting point nh, the chain Φ_h is "stochastically smaller" than Φ, in the sense that if τ_0^h is the first return time to zero by Φ_h then

$$\mathsf{P}_0(\tau_0^h \le k) \ge \mathsf{P}_0(\tau_0 \le k).$$

Hence Φ is transient if Φ_h is transient.

But now we have that

$$\begin{array}{rcl} \beta_h := \sum_n nh\, \Gamma_h(nh) & \ge & \sum_n \int_{nh}^{(n+1)h} (w-h)\Gamma(dw) \\ & = & \int (w-h)\Gamma(dw) \\ & = & \beta - h \end{array} \tag{9.47}$$

so that if $h < \beta$ then $\beta_h > 0$.

Finally, for such sufficiently small h we have that the chain Φ_h is transient from Proposition 9.1.2, as required. □

Let us next consider the use of stochastic comparison methods for the scalar linear model

$$X_n = \alpha X_{n-1} + W_n.$$

Proposition 9.5.2 *Suppose the increment variable W in the scalar linear model is symmetric with density positive everywhere on $[-R, R]$ and zero elsewhere. Then the scalar linear model is Harris recurrent if and only if $|\alpha| \le 1$.*

PROOF The linear model is, under the conditions on W, a μ^{Leb}-irreducible chain on $\mathbb{R}$ with all compact sets petite.

Suppose $\alpha > 1$. By stochastic comparison of this model with a random walk Φ on a half line with mean increment $\alpha - 1$ it is obvious that provided the starting point $x > 1$, then (9.46) holds with $C = (-\infty, 1]$. Since this set is transient for the random walk, as we have just shown, it must therefore be transient for the scalar linear model. Provided the starting point $x < -1$, then by symmetry, the hitting times on the set $C = [-1, \infty)$ are also infinite with positive probability. This argument does not require bounded increments.

If $\alpha < -1$ then the chain oscillates. If the range of W is contained in $[-R, R]$, with $R > 1$, then by choosing $x > R$ we have by symmetry that the hitting time of the chain $X_0, -X_1, X_2, -X_3, \dots$ on $C = (-\infty, 1]$ is stochastically bounded below by the hitting time of the previous linear model with parameter $|\alpha|$; thus the set $[-R, R]$ is uniformly transient for both models.

Thirdly, suppose that the $0 < \alpha \leq 1$. Then by stochastic comparison with random walk on a half line and mean increment $\alpha - 1$, from $x > R$ we have that hitting time on $[-R, R]$ of the linear model is bounded above by the hitting time on $[-R, R]$ of the random walk; whilst by symmetry the same is true from $x < -R$. Since we know random walk is Harris recurrent it follows that the linear model is Harris recurrent.

Finally, by considering an oscillating chain we have the same recurrence result for $-1 \leq \alpha \leq 0$. □

The points to note in this example are

(i) without some bounds on W, in general it is difficult to get a stochastic comparison argument for transience to work on the whole real line: on a half line, or equivalently if $\alpha > 0$, the transience argument does not need bounds, but if the chain can oscillate then usually there is insufficient monotonicity to exploit in sample paths for a simple stochastic comparison argument to succeed;

(ii) even with $\alpha > 0$, recurrence arguments on the whole line are also difficult to get to work. They tend to guarantee that the hitting times on half lines such as $C = (-\infty, 1]$ are finite, and since these sets are not compact, we do not have a guarantee of recurrence: indeed, for transient oscillating linear systems such half lines are reached on alternate steps with higher and higher probability.

Thus in the case of unbounded increments more delicate arguments are usually needed, and we illustrate one such method of analysis next.

9.5.2 Unrestricted random walk and SETAR models

Consider next the unrestricted random walk on $\mathbb{R}$ given by

$$\Phi_n = \Phi_{n-1} + W_n.$$

This is easy to analyze in the transient situation using stochastic comparison arguments, given the results already proved.

Proposition 9.5.3 *If the mean increment of an irreducible random walk on* $\mathbb{R}$ *is non-zero then the walk is transient.*

PROOF Suppose that the mean increment of the random walk $\boldsymbol{\Phi}$ is positive. Then the hitting time $\tau_{\{-\infty,0\}}$ on $\{-\infty, 0\}$ from an initial point $x > 0$ is the same as the hitting time on $\{0\}$ itself for the associated random walk on the half line; and we have shown this to be infinite with positive probability. So the unrestricted walk is also transient.

The argument if $\beta < 0$ is clearly symmetric. □

This model is non-evanescent when $\beta = 0$, as we showed under a finite variance assumption in Proposition 9.4.5.

Now let us consider the more complex SETAR model

$$X_n = \phi(j) + \theta(j)X_{n-1} + W_n(j), \qquad X_{n-1} \in R_j$$

where $-\infty = r_0 < r_1 < \cdots < r_M = \infty$ and $R_j = (r_{j-1}, r_j]$; recall that for each j, the noise variables $\{W_n(j)\}$ form independent zero-mean noise sequences, and again let $W(j)$ denote a generic variable in the sequence $\{W_n(j)\}$, with distribution Γ_j.

We will see in due course that under a second order moment condition (SETAR3), we can identify exactly the regions of the parameter space where this nonlinear chain is transient, recurrent and so on.

Here we establish the parameter combinations under which transience will hold: these are extensions of the non-zero mean increment regions of the random walk we have just looked at.

As suggested by Figure B.1-Figure B.3 let us call the *exterior* of the parameter space the area defined by

$$\theta(1) > 1 \tag{9.48}$$

$$\theta(M) > 1 \tag{9.49}$$

$$\theta(1) = 1,\ \theta(M) \leq 1,\ \phi(1) < 0 \tag{9.50}$$

$$\theta(1) \leq 1,\ \theta(M) = 1,\ \phi(M) > 0 \tag{9.51}$$

$$\theta(1) < 0,\ \theta(1)\theta(M) > 1 \tag{9.52}$$

$$\theta(1) < 0,\ \theta(1)\theta(M) = 1,\ \phi(M) + \theta(M)\phi(1) < 0 \tag{9.53}$$

In order to make the analysis more straightforward we will make the following assumption as appropriate.

(SETAR3) The variances of the noise distributions for the two end intervals are finite; that is,

$$\mathsf{E}(W^2(1)) < \infty, \qquad \mathsf{E}(W^2(M)) < \infty$$

Proposition 9.5.4 *For the SETAR model satisfying the assumptions (SETAR1)-(SETAR3), the chain is transient in the exterior of the parameter space.*

Proof Suppose (9.49) holds. Then the chain is transient, as we show by stochastic comparison arguments. For until the first time the chain enters $(-\infty, -r_{M-1})$ it follows the sample paths of a model

$$X'_n = \phi(M) + \theta(M)X'_{n-1} + W_M$$

and for this linear model $\mathsf{P}_x(\tau_{(-\infty,0)} < \infty) < 1$ for all sufficiently large x, as in the proof of Theorem 9.5.2, by comparison with random walk.

When (9.48) holds, the chain is transient by symmetry: now we find $\mathsf{P}_x(\tau_{(0,\infty,)} < \infty) < 1$ for all sufficiently negative x.

When (9.52) holds the same argument can be used, but now for the two step chain: the one-step chain undergoes larger and larger oscillations and thus there is a positive probability of never returning to the set $[r_1, r_{M-1}]$ for starting points of sufficiently large magnitude.

Suppose (9.50) holds and begin the process at $x_o < \min(0, r_1)$. Then until the first time the process exits $(-\infty, \min(0, r_1))$, it has exactly the sample paths of a random walk with negative drift, which we showed to be transient in Section 8.5. The proof of transience when (9.51) holds is similar.

We finally show the chain is transient if (9.53) holds, and for this we need (SETAR3). Here we also need to exploit Theorem 8.4.2 directly rather than construct a stochastic comparison argument.

Let a and b be positive constants such that $-b/a = \theta(1) = 1/\theta(M)$. Since $\phi(M) + \theta(M)\phi(1) < 0$ we can choose u and v such that $-a\phi(1) < au + bv < -b\phi(M)$. Choose c positive such that

$$c/a - u > \max(0, r_{M-1}), \qquad -c/b - v < \min(0, r_1).$$

Consider the function

$$V(x) = \begin{cases} 1 - 1/a(x+u), & x > c/a - u \\ 1 - 1/c & -c/b - v < x < c/a - u \\ 1 + 1/b(x+v) & x < -c/b - v \end{cases}$$

Suppose $x > R > c/a - u$, where R is to be chosen. Let

$$\lambda(x) = \phi(M) + \theta(M)x + v$$

and

$$\delta(x) = \phi(M) + \theta(M)x + u.$$

If we write

$$\begin{array}{rcl} V_0(x) & = & -a^{-1}\mathsf{E}[(1/(\delta(x) + W(M)))\mathbb{1}_{[W(M) > c/a - \delta(x)]}] \\ V_1(x) & = & -c^{-1}P(-c/b - \lambda(x) < W(M) < c/a - \delta(x)) \\ V_2(x) & = & 1/a(x+u) + b^{-1}\mathsf{E}[(1/(\lambda(x) + W(M)))(_{[W(M) < -c/b - \lambda(x)]}] \end{array} \tag{9.54}$$

then we get

$$\mathsf{E}_x[V(X_1)] = V(x) + V_0(x) + V_1(x) + V_2(x). \tag{9.55}$$

It is easy to show that both $V_0(x)$ and $V_1(x)$ are $o(x^{-2})$. Since

$$1/(\lambda(x) + W(M)) = 1/\lambda(x) - W(M)/\lambda(x)(\lambda(x) + W(M)),$$

the second summand of $V_2(x)$ equals

$$\mathit{\Gamma}_M(-\infty, -c/b - \lambda(x))/b\lambda(x) - \mathsf{E}[(W(M)/\lambda(x)(\lambda(x) + W(M)))\mathbb{1}_{[W(M) < -c/b - \lambda(x)]}].$$

Since for $0 < W(M) < -c/b - \lambda(x)$

$$1/(1 + W(M)/\lambda(x)) \leq 1 + bW(M)/c$$

we have in this case that for x large enough

$$\begin{array}{rcl} 0 & \geq & -x^2 W(M)/\lambda(x)(\lambda(x) + W(M)) \\ & \geq & -x^2 W(M)(1 + bW(M)/c)/\lambda^2(x) \\ & \geq & -2W(M)(1 + bW(M)/c)/\theta^2(M); \end{array} \tag{9.56}$$

whilst for $W(M) \leq 0$, we have

$$1/(1+W(M)/\lambda(x)) \leq 1$$

and so

$$\begin{aligned} 0 &\leq -x^2 W(M)/\lambda(x)(\lambda(x)+W(M)) \\ &\leq -x^2 W(M)/\lambda^2(x) \\ &\leq -2W(M)/\theta^2(M). \end{aligned} \tag{9.57}$$

Thus, by the Dominated Convergence Theorem,

$$\begin{aligned} \lim x^2 \mathsf{E}[-W(M)/\lambda(x)(\lambda(x) &+ W(M))\mathbb{1}_{[W(M)<-c/b-\lambda(x)]}] \\ &= \mathsf{E}[-W(M)/\theta^2(M)] = 0. \end{aligned} \tag{9.58}$$

From (9.58) we therefore see that V_2 equals

$$1/a(x+u) + 1/b\lambda(x) - \Gamma_M(-c/b - \lambda(x), \infty)/b\lambda(x) - o(x^{-2})$$

$$= (b\phi(M) + bv + au)/ab\lambda(x)(x+u) - o(x^{-2}).$$

We now have from the breakup (9.55) that by choosing R large enough

$$\begin{aligned} \mathsf{E}_x[V(X_1)] &= V(x) + (b\phi(M) + bv + au)/ab\lambda(x)(x+u) - o(x^{-2}) \\ &\geq V(x), \qquad x > R. \end{aligned} \tag{9.59}$$

Similarly, for $x < -R < -c/b - v < r_1$, it can be shown that

$$\mathsf{E}_x[V(X_1)] \geq V(x).$$

We may thus apply Theorem 8.4.2 with the set C taken to be $[-R, R]$, and the test function V above to conclude that the process is transient. □

9.5.3 General chains with bounded increments

One of the more subtle uses of the drift conditions involves a development of the interplay between first and second moment conditions in determining recurrence or transience of a chain.

When the state space is $\mathbb{R}$, then even for a chain $\boldsymbol{\Phi}$ which is not a random walk it makes obvious sense to talk about the increment at x, defined by the random variable

$$W_x = \{\Phi_1 - \Phi_0 \mid \Phi_0 = x\} \tag{9.60}$$

with probability law

$$\Gamma_x(A) = \mathsf{P}(\Phi_1 \in A + x \mid \Phi_0 = x).$$

The defining characteristic of the random walk model is then that the law Γ_x is independent of x, giving the characteristic spatial homogeneity to the model.

In general we can define the "mean drift" at x by

$$m(x) = \mathsf{E}_x[W_x] = \int w\, \Gamma_x(dw)$$

so that $m(x) = \Delta V(x)$ for the special choice of $V(x) = x$.

Let us denote the second moment of the drift at x by

$$v(x) = \mathsf{E}_x[W_x^2] = \int w^2\, \Gamma_x(dw).$$

We will now show that there is a threshold or detailed balance effect between these two quantities in considering the stability of the chain.

For ease of exposition let us consider the case where the increments again have uniformly bounded range: that is, for some R and all x,

$$\Gamma_x[-R, R] = 1. \tag{9.61}$$

To avoid somewhat messy calculations such as those for the random walk or SETAR models above we will fix the state space as $\mathbb{R}_+$ and we will make the assumption that the measures Γ_x give sufficient weight to the negative half line to ensure that the chain is a δ_0-irreducible T-chain and also that $v(x)$ is bounded from zero: this ensures that recurrence means that τ_0 is finite with probability one and that transience means that $\mathsf{P}_0(\tau_0 < \infty) < 1$. The δ_0-irreducibility and T-chain properties will of course follow from assuming, for example, that $\varepsilon < \Gamma_x(-\infty, -\varepsilon)$ for some $\varepsilon > 0$.

Theorem 9.5.5 *For the chain $\boldsymbol{\Phi}$ with increment (9.60) we have*

(i) *if there exists $\theta < 1$ and x_0 such that for all $x > x_0$*

$$m(x) \le \theta v(x)/2x \tag{9.62}$$

then $\boldsymbol{\Phi}$ is recurrent.

(ii) *if there exists $\theta > 1$ and x_0 such that for all $x > x_0$*

$$m(x) \ge \theta v(x)/2x \tag{9.63}$$

then $\boldsymbol{\Phi}$ is transient.

Proof **(i)** We use Theorem 9.1.8, with the test function

$$V(x) = \log(1 + x), \qquad x \ge 0: \tag{9.64}$$

for this test function (V1) requires

$$\int_{-x}^{\infty} \Gamma_x(dw)[\log(w + x + 1) - \log(x + 1)] \le 0, \tag{9.65}$$

and using the bounded range of the increments, the integral in (9.65) after a Taylor series expansion is, for $x > R$,

$$\begin{aligned} \int_{-R}^{R} \Gamma_x(dw)[w/(x+1) - w^2/2(x+1)^2 + o(x^{-2})] = \\ m(x)/(x+1) - v(x)/2(x+1)^2 + o(x^{-2}). \end{aligned} \tag{9.66}$$

If $x > x_0$ for sufficiently large $x_0 > R$, and $m(x) \le \theta v(x)/2x$, then

$$\int P(x, dy)V(y) \le V(x)$$

and hence from Theorem 9.1.8 we have that the chain is recurrent.

(ii) It is obvious with the assumption of positive mean for Γ_x that for any x the sets $[0, x]$ and $[x, \infty)$ are both in $\mathcal{B}^+(\mathsf{X})$.

In order to use Theorem 9.1.8, we will establish that for some suitable monotonic increasing V

$$\int_y P(x, dy)V(y) \ge V(x) \tag{9.67}$$

for $x \ge x_0$. An appropriate test function in this case is given by

$$V(x) = 1 - [1 + x]^{-\alpha}, \qquad x \ge 0: \tag{9.68}$$

we can write (9.67) for $x > R$ as

$$\int_{-R}^{R} \Gamma_x(dw)[(w + x + 1)^{-\alpha} - (x + 1)^{-\alpha}] \ge 0. \tag{9.69}$$

Applying Taylor's Theorem we see that for all w we have that the integral in (9.69) equals

$$\alpha m(x)/(x + 1)^{1+\alpha} - \alpha v(x)/2(x + 1)^{2+\alpha} + O(x^{-3-\alpha}). \tag{9.70}$$

Now choose $\alpha < \theta - 1$. For sufficiently large x_0 we have that if $x > x_0$ then from (9.70) we have that (9.69) holds and so the chain is transient. □

The fact that this detailed balance between first and second moments is a determinant of the stability properties of the chain is not surprising: on the space $\mathbb{R}_+$ all of the drift conditions are essentially linearizations of the motion of the chain, and virtually independently of the test functions chosen, a two term Taylor series expansion will lead to the results we have described.

One of the more interesting and rather counter-intuitive facets of these results is that it is possible for the first-order mean drift $m(x)$ to be positive and for the chain to still be recurrent: in such circumstances it is the occasional negative jump thrown up by a distribution with a variance large in proportion to its general positive drift which will give recurrence.

Some weakening of the bounded range assumption is obviously possible for these results: the proofs then necessitate a rather more subtle analysis and expansion of the integrals involved. By choosing the iterated logarithm

$$V(x) = \log\log(x + c)$$

as the test function for recurrence, and by more detailed analysis of the function

$$V(x) = 1 - [1 + x]^{-\alpha}$$

as a test for transience, it is in fact possible to develop the following result, whose proof we omit.

Theorem 9.5.6 *Suppose the increment W_x given by (9.60) satisfies*

$$\sup_x \mathsf{E}_x[|W_x|^{2+\varepsilon}] < \infty$$

for some $\varepsilon > 0$. Then

(i) *if there exists* $\delta > 0$ *and* x_0 *such that for all* $x > x_0$

$$m(x) \leq v(x)/2x + O(x^{-1-\delta}) \tag{9.71}$$

the chain $\boldsymbol{\Phi}$ *is recurrent.*

(ii) *if there exists* $\theta > 1$ *and* x_0 *such that for all* $x > x_0$

$$m(x) \geq \theta v(x)/2x \tag{9.72}$$

then $\boldsymbol{\Phi}$ *is transient.* □

The bounds on the spread of Γ_x may seem somewhat artifacts of the methods of proof used, and of course we well know that the zero-mean random walk is recurrent even though a proof using an approach based upon a drift condition has not yet been developed to our knowledge.

We conclude this section with a simple example showing that we cannot expect to drop the higher moment conditions completely.

Let $\mathsf{X} = \mathbb{Z}_+$, and let

$$P(x, x+1) = 1 - c/x, \qquad P(x, 0) = c/x, \qquad x > 0$$

with $P(0, 1) = 1$.

Then the chain is easily shown to be recurrent by a direct calculation that for all $n > 1$

$$\mathsf{P}_0(\tau_0 > n) = \prod_{x=1}^{n} [1 - c/x].$$

But we have $m(x) = -c + 1 - c/x$ and $v(x) = cx + 1 - c/x$ so that

$$2xm(x) - v(x) = (2 - 3c)x^2 - (c+1)x + c$$

which is clearly positive for $c < 2/3$: hence if Theorem 9.5.6 were applicable we should have the chain transient.

Of course, in this case we have

$$\mathsf{E}_x[|W_x|^{2+\varepsilon}] = x^{2+\varepsilon}c/x + 1 - c/x > x^{1+\varepsilon}$$

and the bound on this higher moment, required in the proof of Theorem 9.5.6, is obviously violated.

9.6 Commentary

Harris chains are named after T.E. Harris who introduced many of the essential ideas in [95]. The important result in Theorem 9.1.3, which enables the properties of Q to be linked to those of L, is due to Orey [207], and our proof follows that in [208]. That recurrent chains are "almost" Harris was shown by Tuominen [268], although the key links between the powerful Harris properties and other seemingly weaker recurrence properties were developed initially by Jain and Jamison [106].

We have taken the proof of transience for random walk on $\mathbb{Z}$ using the Strong Law of Large Numbers from Spitzer [255].

Non-evanescence is a common form of recurrence for chains on $\mathbb{R}^k$: see, for example, Khas'minskii [134]. The links between evanescent and transient chains, and the equivalence between Harris and non-evanescent chains under the T-chain condition, are taken from Meyn and Tweedie [178], who proved Theorem 9.2.2. Most of the connections between neighborhood and global behavior of chains are given by Rosenblatt [228, 229] and Tuominen and Tweedie [269].

The criteria for non-evanescence or Harris recurrence here are of course closely related to those in the previous chapter. The martingale argument for non-evanescence is in [178] and [276], but can be traced back in essentially the same form to Lamperti [151]. The converse to the recurrence criterion under the Feller condition, and the fact that it does not hold in general, are new: the construction of the converse function V is however based on a similar result for countable chains, in Mertens et al [168].

The term "norm-like" to describe functions whose sublevel sets are precompact is new. The justification for the terminology is that norm-like functions do, in most of our contexts, measure the distance from a point to a compact "center" of the state space. This will become clearer in later chapters when we see that under a suitable drift condition, the mean time to reach some compact set from $\Phi_0 = x$ is bounded by a constant multiple of $V(x)$. Hence $V(x)$ bounds the mean "distance" to this compact set, measured in units of time. Beneš in [19] uses the term *moment* for these functions. Since "moments" are standard in referring to the expectations of random variables, this terminology is obviously inappropriate here.

Stochastic comparison arguments have been used for far too long to give a detailed attribution. For proving transience, in particular, they are a most effective tool. The analysis we present here of the SETAR model is essentially in Petruccelli et al [214] and Chan et al [43].

The analysis of chains via their increments, and the delicate balance required between $m(x)$ and $v(x)$ for recurrence and transience, is found in Lamperti [151]; see also Tweedie [276]. Growth models for which $m(x) \geq \theta v(x)/2x$ are studied by, for example, Kersting (see [133]), and their analysis via suitable renormalization proves a fruitful approach to such transient chains.

It may appear that we are devoting a disproportionate amount of space to unstable chains, and too little to chains with stability properties. This will be rectified in the rest of the book, where we will be considering virtually nothing but chains with ever stronger stability properties.