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Exercise 13.4.6. Let $\mathcal{G}$ be a sub-σ-algebra, and let X and Y be two independent random variables. Prove by example that $E(XY|\mathcal{G}) \neq E(X|\mathcal{G}).E(Y|\mathcal{G})$.

Solution. Let $(\Omega, \mathcal{F}, P)$ be the Lebesgue measure on $[0, 1]$. Consider the two independent events $A = [0, \frac{3}{9}],\ B = [\frac{2}{9}, \frac{5}{9}]$. Then, $P(A) = P(B) = \frac{1}{3}, P(A \cap B) = \frac{1}{9},\ \ P(A \cup B) = \frac{5}{9}$ and for $C = (A \cup B)^c$ we have: $P(C) = \frac{4}{9}$.

Next, consider the sub-$\sigma-$algebra $\mathcal{G} = \{\phi, C, C^c, \Omega\}$. Then, for any random variable X:

$$\begin{aligned} E(X|\mathcal{G}) &: \Omega \to \mathbb{R} \\ E(X|\mathcal{G})(w) &= E(X|C).1_C(w) + E(X|C^c).1_{C^c}(w). \quad (*) \end{aligned}$$

In particular, for $X = 1_U$, we have:

$$\begin{aligned} P(U|\mathcal{G}) &: \Omega \to \mathbb{R} \\ P(U|\mathcal{G})(w) &= P(U|C).1_C(w) + P(U|C^c).1_{C^c}(w). \quad (**) \end{aligned}$$

Accordingly, for $X = 1_A, Y = 1_B$ $(X.Y = 1_{A \cap B})$, by three applications of $(**)$ we have:

$$\begin{aligned} E(XY|\mathcal{G})(w) &= P(A \cap B|C).1_C(w) + P(A \cap B|C^c).1_{C^c}(w) \\ &= 0 + \frac{P(A \cap B)}{P(A \cup B)}.1_{C^c}(w) = \frac{1/9}{5/9}.1_{C^c}(w) \\ &= \frac{1}{5}1_{C^c}(w), \\ E(X|\mathcal{G})(w) &= P(A|C).1_C(w) + P(A|C^c).1_{C^c}(w) \\ &= 0 + \frac{P(A)}{P(A \cup B)}.1_{C^c}(w) = \frac{1/3}{5/9}.1_{C^c}(w) \\ &= \frac{3}{5}1_{C^c}(w), \\ E(Y|\mathcal{G})(w) &= P(B|C).1_C(w) + P(B|C^c).1_{C^c}(w) \\ &= 0 + \frac{P(B)}{P(A \cup B)}.1_{C^c}(w) = \frac{1/3}{5/9}.1_{C^c}(w) \\ &= \frac{3}{5}1_{C^c}(w), \end{aligned}$$

yielding:

$$E(XY|\mathcal{G}) = \frac{1}{5}1_{C^c} \neq \frac{9}{25}1_{C^c} = (\frac{3}{5}1_{C^c}).(\frac{3}{5}1_{C^c}) = E(X|\mathcal{G}).E(Y|\mathcal{G}).$$

□