Exercise 13.4.6. Let $\mathcal{G}$ be a sub-$\sigma$-algebra, and let $X$ and $Y$ be two independent random variables. Prove by example that $E(XY|\mathcal{G}) \neq E(X|\mathcal{G}).E(Y|\mathcal{G})$.

Solution. Let $(\Omega, \mathcal{F}, P)$ be the Lebesgue measure on $[0,1]$. Consider the two independent events $A = [0,\frac{3}{8}]$, $B = [\frac{2}{5}, \frac{5}{7}]$. Then, $P(A) = P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{7}$, $P(A \cup B) = \frac{5}{7}$ and for $C = (A \cup B)^c$ we have: $P(C) = \frac{2}{7}$.

Next, consider the sub-$\sigma$–algebra $\mathcal{G} = \{\phi, C, C^c, \Omega\}$. Then, for any random variable $X$:

$$
E(X|\mathcal{G}) : \Omega \to \mathbb{R}
$$

$$
E(X|\mathcal{G})(w) = E(X|C).1_C(w) + E(X|C^c).1_{C^c}(w). \quad (*)
$$

In particular, for $X = 1_U$, we have:

$$
P(U|\mathcal{G}) : \Omega \to \mathbb{R}
$$

$$
P(U|\mathcal{G})(w) = P(U|C).1_C(w) + P(U|C^c).1_{C^c}(w). \quad (**)$$

Accordingly, for $X = 1_A, Y = 1_B$ ($X.Y = 1_{A \cap B}$), by three applications of (**) we have:

$$
E(XY|\mathcal{G})(w) = P(A \cap B|C).1_C(w) + P(A \cap B|C^c).1_{C^c}(w)
$$

$$
= 0 + \frac{P(A \cap B)}{P(A \cup B)}1_{C^c}(w) = \frac{1}{9/5}1_{C^c}(w)
$$

$$
= \frac{5}{9}1_{C^c}(w),
$$

$$
E(X|\mathcal{G})(w) = P(A|C).1_C(w) + P(A|C^c).1_{C^c}(w)
$$

$$
= 0 + \frac{P(A)}{P(A \cup B)}1_{C^c}(w) = \frac{1}{3/5}1_{C^c}(w)
$$

$$
= \frac{5}{3}1_{C^c}(w),
$$

$$
E(Y|\mathcal{G})(w) = P(B|C).1_C(w) + P(B|C^c).1_{C^c}(w)
$$

$$
= 0 + \frac{P(B)}{P(A \cup B)}1_{C^c}(w) = \frac{1}{3/5}1_{C^c}(w)
$$

$$
= \frac{5}{3}1_{C^c}(w),
$$

yielding:

$$
E(XY|\mathcal{G}) = \frac{1}{5}1_{C^c} \neq \frac{9}{25}1_{C^c} = (\frac{3}{5}1_{C^c}) (\frac{3}{5}1_{C^c}) = E(X|\mathcal{G}).E(Y|\mathcal{G}).
$$

$\square$