A Detailed Proof of Starting Point Cutoff for a Toy Example

In this web appendix we give a detailed proof of the existence of the starting point cutoff phenomenon for a toy Markov chain. This Markov chain was used as an example in [1] for a different purpose. Consider the Markov kernel defined on \( \mathbb{R}^d \) by

\[
P(x, \cdot) \sim \text{Normal} \left( x, \frac{3}{4} I_d \right)
\]

where \( I_d \) is the \( d \)-dimensional identity matrix. One can show that \( P \) is a regular Markov kernel with stationary distribution \( \pi \sim \text{Normal}(0, I_d) \). In fact one can compute that

\[
P^t(x, \cdot) \sim \text{Normal} \left( x, (I - 4^{-t}) I_d \right)
\]

Now let \( x_n \) be a sequence of starting points in \( \mathbb{R}^d \) such that \( \lim_{n \to \infty} \| x_n \| = \infty \). We will show that \( P \) has starting point cutoff at time \( t_n = \log_2(\| x_n \|) \) starting from \( x_n \). First let \( c > 1 \). For \( n \) sufficiently large such that \( \| x_n \| \geq 1 \) we have

\[
d_{x_n}(ct_n) = \left\| P^{\lfloor ct_n \rfloor}(x_n, \cdot) - \pi \right\|_{TV}
\]

\[
\leq \frac{1}{2} D_{KL}(P^{\lfloor ct_n \rfloor}(x_n, \cdot) \| \pi) \quad \text{by Pinsker’s inequality}
\]

\[
= \frac{1}{2} \sqrt{-d \log(1 - 4^{-\lfloor ct_n \rfloor}) - d4^{-\lfloor ct_n \rfloor} + \| x_n \|^2 4^{-\lfloor ct_n \rfloor}} \quad \text{by the formula for KL divergence of Gaussians}
\]

\[
\leq \frac{1}{2} \sqrt{d(4^{-\lfloor ct_n \rfloor} + 16^{-\lfloor ct_n \rfloor}) - d4^{-\lfloor ct_n \rfloor} + \| x_n \|^2 4^{-\lfloor ct_n \rfloor}} \quad \text{by a Taylor series bounds on logarithm}
\]

\[
= \sqrt{\frac{1}{2} \sqrt{d - \| x_n \|^2} \left( \frac{1}{2} \right)^{\lfloor ct_n \rfloor}}
\]

\[
\leq \frac{1}{2} \sqrt{d + 1} \| x_n \| \left( \frac{1}{2} \right)^{\| x_n \| - 1}
\]

\[
= 2^{\frac{1}{2} \sqrt{d + 1} \| x_n \|^{1-c}}
\]

therefore \( \lim_{n \to \infty} d_{x_n}(ct_n) = 0 \). Now suppose \( c < 1 \). For \( x \in \mathbb{R}^d \) and \( r > 0 \) define

\[
A_x^r = \{ y \in \mathbb{R}^d : y \cdot x \leq r \| x \| \}
\]
Then for any $x \in \mathbb{R}^d$ and $r > 0$

$$\pi(A^t_r) = \mathbb{P}[W \cdot x \leq r\|x\|] \quad \text{where } W \sim \text{Normal}(0, I_d)$$

$$ = \mathbb{P}[Z \leq r] \quad \text{where } Z \sim \text{Normal}(0, 1)$$

$$ = \Phi(r)$$

and for $x \in \mathbb{R}^d, r > 0$ and $t \in \mathbb{N}$

$$P^t(x, A^t_r) = \mathbb{P}\left[\left(\frac{x}{2^t} + \left(\sqrt{1 - 4^{-t}}\right)W\right) \cdot x \leq r\|x\|\right] \quad \text{where } W \sim \text{Normal}(0, I_d)$$

$$ = \mathbb{P}\left[Z \leq \frac{\left(r2^t - \|x\|\right)}{\sqrt{4^t - 1}}\right] \quad \text{where } Z \sim \text{Normal}(0, 1)$$

$$ = \Phi\left(\frac{r2^t - \|x\|}{\sqrt{4^t - 1}}\right)$$

Now define a sequence $r_n = \|x_n\|^{\frac{1-c}{2}}$. Then for each $n \in \mathbb{N}$,

$$d_{x_n}(ct_n) = \text{CPV version(Copy)}[P^{[ct_n]}(x_n, \cdot) - \pi]_{TV}$$

$$\geq |P^{[ct_n]}(x_n, A^t_{x_n}) - \pi(A^t_{x_n})|$$

$$ = \left|\Phi\left(\frac{r_n2^{[ct_n]} - \|x_n\|}{\sqrt{4^{[ct_n]} - 1}}\right) - \Phi(r_n)\right|$$

but since

$$\lim_{n \to \infty} \Phi(r_n) = 1$$

and

$$\frac{r_n2^{[ct_n]} - \|x_n\|}{\sqrt{4^{[ct_n]} - 1}} \leq \|x_n\|^{\frac{1-c}{2}} - \|x_n\|^{1-c}$$

implies

$$\lim_{n \to \infty} \Phi\left(\frac{r_n2^{[ct_n]} - \|x_n\|}{\sqrt{4^{[ct_n]} - 1}}\right) = 0$$

we have that

$$\lim_{n \to \infty} d_{x_n}(ct_n) = 1$$

This completes the proof that $P$ has starting point cutoff at time $t_n = \log_2(\|x_n\|)$ starting from $x_n$.

**References**