Time to $L^2$ for certain random walks on compact Lie groups*

(Notes in progress, 1994.)

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On the unitary group $U(N)$, consider the random walk with step distribution given by the pushforward of the measure $C_a (\sin(\theta/2))^a d\theta \times d\lambda$ under the map $(x, \theta) \mapsto x^{-1} \text{diag}(e^{i\theta}, 1, \ldots, 1)x$, where $x \in U(N)$, $0 \leq \theta < 2\pi$, $\lambda$ is normalized Haar measure on $U(N)$, and $C_a = \left( \int_0^{2\pi} (\sin(\theta/2))^a d\theta \right)^{-1}$. We take $a$ to be an integer between 0 and $N - 1$.

Let $\mu_k$ be the distribution of this random walk after $k$ steps (where the starting distribution $\mu_0$ is a point mass at the identity element of $U(N)$). We are interested in the convergence of $\mu_k$ to $\lambda$.

The case $a = N - 1$ was studied in Porod’s thesis, and it was proved that $\|\mu_k - \lambda\|_{L^2(\lambda)} \leq Ae^{-Bc}$ when $k = \frac{1}{2} N \log N + cN$, where $A$ and $B$ are positive constants. Then since $||\mu_k - \lambda||_{T.V.} \leq \frac{1}{2} ||\mu_k - \lambda||_{L^2(\lambda)}$, her results implied convergence rates (in fact a cut-off phenomenon!) in total variation distance.

We have now observed that when $a = N - 1$, $\mu_k$ in fact has a density in $L^2(\lambda)$ for $k \geq O(N)$.

Porod also showed that when $a = 0$, $||\mu_k - \lambda||_{L^2(\lambda)}$ was infinite for $k < \frac{1}{2} (N^2 - N) + 1$. This dramatically different behaviour prompted the present study, whose goal is to understand the $L^2$ convergence for intermediate values of $a$. In particular, we are interested in conditions on $k$ as a function of $a$ and $N$ which would guarantee that $\mu_k$ is in $L^2(\lambda)$.

Using Fourier analysis and computing characters in a manner similar to the computation in Porod’s thesis, we have shown that

* Dedicated to the memory of Onion Duck.
\[
\| \mu_k - \lambda \|_{L^2(\lambda)} = K_{a,N,k} \sum_{\lambda_1 < \lambda_2 < \ldots < \lambda_N} \left( \sum_{j=1}^{N} (-1)^j \frac{(-\lambda_j - a + N - 2)}{N - 2 - 2a} \prod_{r=j+1}^{N} (\lambda_r - \lambda_j) \prod_{r=1}^{j-1} (\lambda_j - \lambda_r) \right)^{2k} \times \left( \prod_{1 \leq r < s \leq N} (\lambda_s - \lambda_r) \right)^2 - 1,
\]

where \( K_{a,N,k} \) is an explicit constant depending on \( a, N \) and \( k \). Here the sum is taken over all \( N \)-tuples of (positive or negative) integers \((\lambda_1, \lambda_2, \ldots, \lambda_N)\) satisfying \( \lambda_1 < \lambda_2 < \ldots < \lambda_N \), and \( (-\lambda_j - a + N - 2) \) is a binomial coefficient.

We have further shown (by considering the sum over \( m \) of terms with \((\lambda_1, \ldots, \lambda_N) = (-m, m, 2m, \ldots, (N-1)m)\)) that this sum is infinite for \( k < (N^2 - N + 1)/2(a+1) \). (That is, \( \mu_k \) is not a measure in \( L^2(\lambda) \) for this range of \( k \).)

(Andrey Feuerverger has now obtained similar lower bounds by related methods.)

On the other hand, a remark by Gerard Letac made us realize that, since the above measures are mutually absolutely continuous for different values of \( a \), therefore since for \( a = N - 1 \) the measure \( \mu_k \) is absolutely continuous with respect to \( \lambda \) for \( k \geq O(N) \), therefore this same is true for any value of \( a \).

The difficulties with further estimating the sum are that in the inside alternating sum, the individual terms may be going to infinite for large values of the \( \lambda_j \), even though we know that the total value of the alternating sum is bounded by a constant. This makes analysis of the sum extremely sensitive.

One idea we had was to use spherical coordinates for \((\lambda_1, \ldots, \lambda_N)\), and to approximate the sum by an integral. Since we were only interested in the finiteness of the sum, we need only consider those \((\lambda_1, \ldots, \lambda_n)\) sufficiently far from the origin.