WEB APPENDIX FOR:

The Coupling/Minorization/Drift Approach to
Markov Chain Convergence Rates

by (in alphabetical order)
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This web appendix provides proofs of the computational lemmas in the main article, which is available at: www.probability.ca/NoticesArt.pdf

Proof of Lemma 1:

To avoid problematic configurations where the particles are very close together, we first set $X' = \{(x_1, x_2, x_3) \in X : \forall 1 \leq i < j \leq 3, |x_i - x_j| \geq 1/4\}$. Since $X'$ is a compact set, and $\pi$ is continuous and positive on $X'$, it must achieve its minimum $m := \min_{x,y \in X'} \frac{\pi(y)}{\pi(x)} > 0$ there. Let $A \subset X$. Then from any state $x \in X$, the chain will move into $A$ on the next step provided that the proposed new configuration $y$ is within the subset $A$, and that the proposal is accepted. Hence,

$$P(x,A) = \int_A P(x,dy) = \int_A \min[1, \frac{\pi(y)}{\pi(x)}] dy \geq \int_{A \cap X'} m dy = m \text{Leb}(A \cap X'),$$

where Leb is Lebesgue measure on $\mathbb{R}^6$. So, if we set $\epsilon = m \text{Leb}(X')$, and $\nu(A) = \text{Leb}(A \cap X') / \text{Leb}(X')$, then $\epsilon > 0$, and $\nu$ is a probability measure, and $P(x,A) \geq \epsilon \nu(A)$, i.e. a uniform minorization condition is satisfied.

To obtain quantitative convergence bounds, we need to estimate $\text{Leb}(X')$ and $m$. In order for $(x_1, x_2, x_3) \in X'$, we can choose any $x_1 \in [0,1]^2$ (with two-dimensional area 1), then choose any $x_2 \in [0,1]^2 \setminus B(x_1, 1/4)$ (with area $\geq 1 - 3.14(1/4)^2$), then choose any $x_3 \in [0,1]^2 \setminus (B(x_1, 1/4) \cup B(x_2, 1/4))$ (with area $\geq 1 - 3.14(1/4)^2 - 3.14(1/4)^2$). [Here $B(x,r)$ is the two-dimensional disc centered at $x$ of radius $r$, with area $3.14 r^2$, where we write the constant as “3.14” to avoid confusion with the stationary distribution $\pi(\cdot)$.] Hence, $\text{Leb}(X') \geq (1)(1 - \frac{3.14}{16})(1 - \frac{3.14}{8}) \geq 0.48$.

Also, for any $x \in X'$, we must have $0 \leq |x_i| \leq \sqrt{2}$ and $1/4 \leq |x_i - x_j| \leq \sqrt{2}$, so therefore

$$0 \leq |x_1| + |x_2| + |x_3| \leq 3\sqrt{2}, \quad \text{and} \quad \frac{3}{\sqrt{2}} \leq \sum_{i<j} |x_i - x_j|^{-1} \leq 12.$$
It follows that
\[
m \geq \frac{e^{-C(3\sqrt{2})-D(12)}}{e^{-C(0)-D(3/\sqrt{2})}} = e^{-C(3\sqrt{2})-D(12-3/\sqrt{2})} \geq e^{-C(4.25)-D(7.76)}.
\]

Hence,
\[
\epsilon = m \text{ Leb}(X') \geq (0.48) \cdot e^{-C(4.25)-D(7.76)},
\]
as claimed.

**Proof of Lemma 2:**

Let \( x \in C \). Without loss of generality, assume \( x \geq 0 \). First consider \( B \subset [-1, 1] \), and let \( z \in [0, 1] \) and \( y \in B \). Then we must have \([0, 1] \subseteq [x-2, x+2] \), and \( B \subseteq [z-2, z+2] \). Hence, the proposal density \( q \) satisfies that \( q(x, z) = q(z, y) = \frac{1}{4} \).

Also, \( \pi(x) \leq e^0 = 1 \), and \( e^{-1} \leq \pi(y) \leq 1 \), and \( \pi(z) \geq e^{-1} \), so if \( \alpha(x, z) = \min[1, \frac{\pi(z)}{\pi(x)}] \) is the probability of accepting a proposed move from \( x \) to \( z \), then \( \alpha(x, z) \geq e^{-1} \) and \( \alpha(z, y) \geq e^{-1} \). Hence,
\[
P^2(x, B) \geq \int_B \int_{x-2}^{x+2} q(x, z) \alpha(x, z) q(z, y) \alpha(z, y) \, dz \, dy \\
\geq \int_B \int_0^1 (1/4)(e^{-1})(1/4)(e^{-1}) \, dz \, dy = \frac{1}{16e^2} \text{ Leb}(B).
\]

Finally, for any \( A \subseteq \mathbb{R} \),
\[
P^2(x, A) \geq P^2(x, A \cap [-1, 1]) \geq \frac{1}{16e^2} \text{ Leb}(A \cap [-1, 1]) = \frac{1}{8e^2} \nu(A),
\]
which gives the result.

**Proof of Lemma 3:**

Without loss of generality, assume \( x \geq 0 \). Note that
\[
PV(x) = \int_{x-2}^{x+2} q(x, y) [V(y)\alpha(x, y) + V(x)(1 - \alpha(x, y))] \, dy.
\]

We first compute the “top half” of this integral, where \( x \leq y \leq x - 2 \). Here
\[
\alpha(x, y) = \frac{\pi(y)}{\pi(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}, \text{ and } q(x, y) = 1/4, \text{ so }
\]
\[
\int_x^{x+2} q(x, y) [V(y)\alpha(x, y) + V(x)(1 - \alpha(x, y))] \, dy
\]
\[
= \int_x^{x+2} \frac{1}{4} e^{\frac{x}{2}} e^{x-y} dy + \int_x^{x+2} \frac{1}{4} e^{\frac{x}{2}} (1 - e^{x-y}) dy
\]
\[
= \frac{1}{4} e^{x} \int_x^{x+2} e^{-\frac{y}{2}} dy + \frac{1}{4} e^{\frac{x}{2}} (2) - \frac{1}{4} e^{\frac{3x}{2}} \int_x^{x+2} e^{-y} dy
\]
\[
= \frac{1}{4} e^{x} \left[-2e^{-\frac{x+2}{2}} + 2e^{-\frac{x}{2}}\right] + \frac{1}{4} e^{\frac{x}{2}} (2) - \frac{1}{4} e^{\frac{3x}{2}} [-e^{-x-2} + e^{-x}]
\]
\[
= \frac{1}{4} e^{\frac{x}{2}} (-2e^{-1} + 2 + e^{-2} - 1)
\]
\[
= \frac{1}{4} (3 + e^{-2} - 2e^{-1}) V(x) \equiv \lambda_1 V(x),
\]

where \(\lambda_1 = \frac{1}{4}(3 + e^{-2} - 2e^{-1}) \approx 0.6. \) Then we consider three different cases:

Case 1: \(x \in (2, \infty) \subseteq C = [-2, 2]. \) Then \(\alpha(x, y) := \min\{1, \frac{e^{-|y|}}{e^{-x}}\} = 1 \) for all \(y \in [x-2, x), \)

\[
PV(x) = \int_{x-2}^{x} q(x, y)V(y)dy + \lambda_1 V(x) = \frac{1}{4} \int_{x-2}^{x} e^{\frac{y}{2}} dy + \lambda_1 V(x)
\]
\[
= \frac{1}{4} e^{\frac{x}{2}} 2(1 - e^{-1}) + \lambda_1 V(x) = \left(\frac{1}{2} (1 - e^{-1}) + \lambda_1\right)V(x) \leq 0.916 V(x).
\]

Case 2: \(x \in [1, 2] \subseteq C. \) Again \(\alpha(x, y) = 1 \) for all \(y \in [x-2, x), \)

\[
PV(x) = \int_{x-2}^{x} V(y)q(x, y)dy + \lambda_1 V(x) = \frac{1}{4} \left(\int_{x-2}^{x} e^{-\frac{y}{2}} dy + \int_{0}^{x} e^{\frac{y}{2}} dy\right) + \lambda_1 V(x)
\]
\[
= \frac{1}{4} (\int_{0}^{2-x} e^{\frac{y}{2}} dy + \int_{0}^{x} e^{\frac{y}{2}} dy) + \lambda_1 V(x) = \frac{1}{2} (e^{\frac{x}{2}} + e^{1 - \frac{x}{2}}) - 1 + \lambda_1 e^{\frac{x}{2}}
\]

Let \(z = e^{\frac{x}{2}}. \) Then, computing numerically,

\[
\max_{x \in [1, 2]} [PV(x) - 0.916 V(x)] = \max_{z \in [\sqrt{e}, e]} \left[\frac{1}{2} (z + e^{-\frac{z}{2}}) - 1 + \lambda_1 z - 0.916 z\right] \leq 0.13.
\]

Case 3: \(x \in [0, 1] \subseteq C. \) Then \(\alpha(x, y) = 1 \) for any \(y \in [-x, x]. \)

\[
PV(x) = \int_{x-2}^{-x} q(x, y)\alpha(x, y)V(y) + q(x, y)(1 - \alpha(x, y))V(x)\] \, dy
\]
\[
+ \int_{-x}^{x} q(x, y)V(y) dy + \lambda_1 V(x)
\]
\[
= \frac{1}{4} e^{\frac{x}{2}} \int_{x}^{2-x} (e^{\frac{x-y}{2}} + 1 - e^{x-y}) dy + \frac{1}{2} \int_{0}^{x} e^{\frac{y}{2}} dy + \lambda_1 e^{\frac{x}{2}}
\]
\[
= \frac{e^{\frac{x}{2}}}{4} \left[-2e^{x-1} + 2(e^{x-1}) - 2x + 3\right] + e^{\frac{x}{2}} - 1 + \lambda_1 e^{\frac{x}{2}}.
\]
Computing numerically, this implies that

$$\max_{x \in [0, 1]} [PV(x) - 0.916 V(x)] \leq 0.285.$$  

Combining these three cases (and their symmetric versions for $x < 0$) shows that the univariate drift condition

$$PV(x) \leq 0.916 V(x) + 0.285 1_C(x)$$

holds for all $x \in \mathcal{X}$, as claimed.