Active-Learning Strategies in Advanced Mathematics Classes

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Abstract. Advanced mathematics and other theoretical sciences are often taught purely using the lecture format, which promotes passivity and isolation in students. This paper advocates the use of certain alternative teaching methods, including cooperative learning in small groups and essay-writing assignments about technical topics. The emphasis throughout is on getting students to participate more, to interact more, and to broaden their perspective. Efforts are described to implement these reforms in upper-division courses in probability theory at the University of Minnesota, and students' anonymous questionnaires are analyzed. It is argued that many of these teaching reforms can be implemented quite generally, without huge effort, and without greatly disrupting the normal flow of a course.

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I hear, I forget;
I see, I remember;
I do, I understand.

– old proverb

Introduction.

It is generally recognized that there are a number of shortcomings in the traditional lecture approach to the teaching of mathematics (and similarly other theoretical sciences such as statistics, physics, computer science, and engineering). Most mathematicians agree that the best way to learn mathematics is by actively *doing* mathematics; by discussing it with others; and by synthesizing major ideas. However, in typical university mathematics classes in the United States, students passively watch a professor lecture at a blackboard. They seldom speak in class; often don't get to know a single fellow student; don't synthesize or expand upon the material; and are relegated to solving narrow, computational problems, working individually. What knowledge they do acquire is often learned by "rote" and does not reflect deep understanding. The students may attend a "recitation session" of some sort to discuss solutions to homework problems, but such sessions are typically of limited success and are often considered secondary. It is understandable, then, that students often don't see mathematics as the dynamic, exciting, creative discipline that it is.

This situation is cause for concern. Many potentially successful students become uninterested in mathematics, and fail to learn it well or to enroll in subsequent courses. Furthermore, studies indicate (Light, 1990) that women are particularly affected in this way, so that traditional teaching practices may partially account for the small numbers of successful female mathematics students.

The purpose of this article is to advocate that university mathematics (and related disciplines) teachers experiment (in the spirit of Cross & Angelo, 1988; Angelo, 1991) with augmenting the standard lecture format with certain alternative teaching strategies. Efforts are described to incorporate these strategies in small (about 15 students), advanced courses in probability theory at the University of Minnesota, a large American research university. It is argued that these strategies are simple and straightforward enough that they can be implemented without huge amounts of effort, and without significantly disrupting the normal flow of a course. The practices are often very popular with students. In addition, they can add a whole new level of excitement, challenge, and personal involvement to the teaching process itself.

The changes that we advocate are in keeping with recent systemic trends. For example, in the United States, both the National Council of Teachers of Mathematics (1989) at the pre-university level, and the Calculus reform movement (see for example Hughes-Hallet et al., 1992) at the beginning university level, have called for changes to make the material more relevant to the students' lives and to make the students more active participants in their learning. Part of the claim of this article is that it is not necessary to wait for such large-scale initiatives. Rather, university teachers can implement many changes in their own classroom by their own initiative.

Small-Groups Exercises.

The practice of "cooperative learning", i.e. having students work together in small groups, is often advocated in university teaching (Johnson, Johnson, & Smith, 1991) and for elementary-level mathematics teaching (Johnson & Johnson, 1991). It has more recently been advocated for teaching university-level statistics (Jones, 1991; Mosteller, 1989). It is known to be beneficial to students in a number of ways. Students are better able to learn and retain concepts when they are actively involved; students can learn *from* each other, and can learn from *teaching* each other; students can get practice working and communicating with others (an essential skill in many job settings); and students sense a warmer, more welcoming, and more caring atmosphere. Research indicates that this last benefit is especially important for the retention and success of female students (see e.g. Light, 1990). Also, failure to work together may be a major cause of low achievement among minority students (Treisman, 1992). However, cooperative learning is rarely utilized in the teaching of advanced mathematics, and we approached it with some trepidation.

In our probability courses, the students were organized into groups of between 3 and 5 students, and were assigned problem-solving exercises which typically lasted one or two class hours. The instructor was available only for questions and for providing assistance or encouragement if a group appeared to be having difficulties or not working together well. The exercises were held approximately once every two weeks, as a supplement to the usual lectures. Thus, the pace and overall structure of the course was not greatly affected.

Despite our initial hesitations, the small-group exercises appeared to be quite successful. The students usually participated enthusiastically, and seriously discussed the material among themselves. There was also a carry-over effect: once students were used to working together, they would sometimes discuss course material outside of class time.

The (anonymous) student feedback was also quite positive, and got progressively more so. At the end of the second quarter, fully 7 out of 12 student respondents gave the small-groups exercises a rating of "5" (best), 2 students gave them a rating of "4", and the remaining 3 students gave them a rating of “3” (neutral). No student gave the small-group exercises a negative rating. Written comments included such statements as “Small group exercises are a great help” and “The student input and discussion help my understanding of material.” Incidentally, most students also indicated that it was not necessary for the exercises to be graded.

Small-groups exercises are not without their problems. It typically takes the groups a fair bit of time to get started, to communicate with each other, and to present solutions if requested. This obviously takes time away from the lectures. However, this difficulty can be partially alleviated by putting the group exercises to good use, for example to review a topic so that it doesn’t have to be reviewed extensively in lecture.

Another problem is that most mathematics students are often not skilled at working with other people, and a group may splinter off into a collection of individuals thinking separately, making the weaker students feel abandoned. This did not happen to any great extent, but it is difficult to overcome (though timely comments from the instructor can be of help).

Despite these difficulties, our experience with cooperative learning techniques has been overwhelmingly positive. They do not take particularly large effort to implement (at least in a moderately sized class), and they are popular and productive.

Writing to Learn.

Written assignments are known to be a good way to get students to think about what they are learning, to remember more, and to see course material in a larger context. They have been advocated across the curriculum, including in mathematics (Connolly & Vilardi, 1989). However, like small-group exercises, they are only rarely used in higher mathematics classes, and again we approached them with some trepidation.

Our students were asked to write short (about 5 pages typed double-spaced) essays that clearly explained, illustrated, and/or extended, some particular aspect of the course material. Specific suggestions included applying the course material to some other subject (this was particularly appropriate for graduate students from other departments, who were using probability theory in their own research), and running computer simulations to verify some theoretical result presented in the course (this generated a lot of interest).

Students took the assignments quite seriously, with fully 13 out of 16 turning in papers for the first assignment that were considered to be good or excellent, and with similar numbers for the second assignment.

The writing assignments provoked mixed student reviews. Most students acknowledged that they would remember their essay topic for longer than they would remember most of the rest of the course (mean rating of 3.9 out of 5). However, some students found the essays to be awkward and unusual to write. Indeed, after the second quarter, the essay assignment received a mean approval rating of 3.0 out of 5 (i.e. a neutral response). These results are somewhat disappointing, and one hopes that students would feel better about these assignments if they became more accustomed to them. However, balancing the overall neutral rating is the fact that some students (especially lower-achieving students) responded very well to the essay assignments, providing such comments as “I like what you've brought to this course [i.e. the essays and group assignments]”.

Despite the mixed student reviews, we are confident that these writing assignments help the students to broaden their perspective on the course material. The assignments force them to put together several ideas, and to explain them clearly. The students increase their ability to communicate about technical matters, an important skill for many jobs (including pure mathematical research). Also, the students hopefully feel an “attachment” to some specific topic from the course, which they might remember long after most of the course material has faded. Most importantly, the students see that there is more to mathematical thinking than merely solving isolated, computational problems.

Peer Review of Writing Assignments.

It is possible to combine writing assignments with cooperative learning techniques, by having students read and review each other's essay papers (similar to Johnson et al, 1991, 4:18). Hopefully this can help both writer and reader to learn more about the subject, to interact well, and to produce improved essays.

Our students were required to complete their second essays early, and to exchange them with two classmates (chosen from the student's established small group). They were then required to fill out a one-page review form that asked about strengths, weaknesses, and suggestions of their classmates' work. The students later met in their small groups to discuss the essays orally. The review forms were returned to the essay-writers, who were then required to prepare final versions of their essays by the last day of class.

Students did not make extensive changes to their essays as a result of the reviews, though a number of small changes were made. Still, students felt that the reviews helped them to improve their essays; they gave this statement a mean rating of 4.25 out of 5. More important was the extent to which the students appreciated the review process. They gave mean ratings of between 4.4 and 4.6 out of 5 to appreciating each of: receiving peer reviews, providing peer reviews, and receiving the instructor's comments.

Most interesting was the extent to which students' opinions of the essay-writing exercise *improved* during the peer-review phase. Asked both before and after this phase whether they thought the essay assignment was a good idea, the mean rating improved from 3.0 (with $N = 14$, $S^2 = 0.16$) to 3.6 (with $N = 12$, $S^2 = 0.14$) out of 5. Similarly, asked whether writing the essay helped them to understand their topic more deeply, the mean rating improved from 3.3 (with $N = 14$, $S^2 = 0.10$) to 4.1 (with $N = 12$, $S^2 = 0.11$) out of 5. This last increase is statistically significant, with a p-value (using approximate normality) of 0.044. (It should be noted that this analysis is conservative, assuming independence of the "pre" and "post" responses even though they would actually be positively correlated. On the other hand, two students were absent for the "post" responses, and it was assumed that their absence was independent of their opinion.)

Our evidence thus suggests: *on average, students feel that the essay-writing assignment helps them to understand their topic more deeply. Furthermore, they feel this even more strongly after participating in a peer-review process.*

Informal Writing Assignments.

In addition to the essay-writing assignments discussed above, writing can be used in a number of less formal, but very useful, ways. (Indeed, it can be argued (Connolly, 1993) that such informal writing is even more important than formal essays.) Appropriate uses of such writing include:

- *Feedback about the course* (Olmsted, 1991). Typically, a teacher never knows (at least not until after the course is over) the opinions of their students regarding workload, comprehensibility of theories and examples, and preferences of teaching practises. However, a well-written anonymous questionnaire, and a few minutes of class time, can provide a large amount of information. Also, students typically appreciate the opportunity to express their views: Asked whether student feedback forms were a good idea, our students gave a mean rating of 4.3 out of 5.

- *The minute-paper at the end of a lecture* (Wilson, 1986; Mosteller, 1989). This standard technique involves asking students to write down what they felt was the key topic and/or the most confusing topic and/or what they most want to know more about from that day's lecture. It takes a few minutes at the end of the class. It can encourage students to pay attention and to remember more of the lecture, and it provides very useful feedback for the instructor.

- *Meta-cognitive exercises* (Angelo, 1991, 24-25). This technique involves asking students to describe (in full sentences) how they have gone about solving a particular homework problem. This forces them to consider their own thought processes, and may lead them to improve the way they approach problems. It also helps the teacher to understand which concepts are causing confusion. Furthermore, this exercise can be added on to the normal homework exercises, so it does not require additional class time.

Conclusion.

Unlike many social sciences, advanced technical topics do not easily lend themselves to energetic discussions of "controversial issues". They also tend to have long, imposing histories. For this reason, it is sometimes felt that they must be taught solely using a dry, lecture format, even if this format is uninspiring to students.

We do not believe this to be true. Rather, we believe that lectures can be augmented by various techniques that encourage active learning by the students. In this paper, we have argued that there are a number of teaching strategies that are available to make university mathematics (and other) classes less sterile, and more participatory and exciting. We have described our own positive experiences with these practices. We believe that the teaching strategies presented here are straightforward enough that they can be implemented without specialized training, by an individual lecturer. Furthermore, they can be used in virtually any problem-solving type of course, at virtually any university level. We strongly believe that they lead to deeper understanding and better retention of knowledge by the students.

We sincerely hope that teachers in technical disciplines will be inspired to explore, however cautiously, these and other strategies in an effort to improve student learning and to make their classroom environments as exciting as the subjects they teach.

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