The Bieberbach Conjecture

A minor thesis submitted by

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1. Introduction.

Let S denote the set of all univalent (i.e. one-to-one) analytic functions f defined in the disk $|z| < 1$, with $f(0) = 0$ and $f'(0) = 1$. Such functions may be written in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ..., \qquad |z| < 1.$$

One example of a function in S is the Koebe function

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + ..., \qquad |z| < 1.$$

Since the function $z \mapsto \frac{1+z}{1-z}$ is univalent with image the right half plane, we see that $z \mapsto \left(\frac{1+z}{1-z}\right)^2$ is univalent, so $k \in S$, and the image of k is the entire complex plane except for real numbers $\leq -\frac{1}{4}$. In 1916, L. Bieberbach [Bi] conjectured that the Koebe function was maximal with respect to the absolute value of the coefficients of its power series. More precisely, he conjectured the following:

The Bieberbach Conjecture. *For each function $f \in S$, we have $|a_n| \leq n$, for $n = 2, 3, 4, \ldots$. Furthermore, equality occurs for any one n only when f is a rotation of the Koebe function, i.e. when $f(z) = \beta^{-1}k(\beta z)$, for some complex constant β with $|\beta| = 1$.*

The Bieberbach conjecture was proved in 1984 by L. de Branges [**dB1**, **dB2**]; see also [dB3]. The proof was simplified slightly by C.H. FitzGerald and Ch. Pommerenke [FP]. Before presenting a proof, we begin with some history.

Bieberbach proved his conjecture only for n=2. In 1923, K. Löwner [Lö2] developed a representation of functions in S which enabled him to prove the conjecture for n=3. In 1925, J.E. Littlewood [Li] proved that $|a_n| < en$ for all n, where $e = 2.718...$, and in 1951 I.E. Bazilevich [Ba] showed that $|a_n| < en/2+1.51$ for all n. This was improved by I.M. Milin [**Mi1**, **Mi3**] to $|a_n| < 1.243n$. C.H. FitzGerald [Fi] used the Goluzin inequalities [Go] to get $|a_n| < \sqrt{7/6}\, n < 1.081n$, and D. Horowitz [Ho] tightened this to $|a_n| < \left(\frac{209}{140}\right)^{1/6} n < 1.0691n$. In 1955, P.R. Garabedian and M. Schiffer [**GS1**] gave a difficult proof that $|a_4| \leq 4$, and in 1960 Z. Charzyński and M. Schiffer [CS] used the Grunsky inequalities [Gru] to give a more elementary proof. In 1968 and 1969, R.N. Pederson [Pe] and M. Ozawa [Oz] independently proved that $|a_6| \leq 6$. In 1972, R.N. Pederson and M. Schiffer [PS] used a strengthening of the Grunsky inequalities by Garabedian and Schiffer [GS2] to prove $|a_5| \leq 5$. The Bieberbach conjecture was proved long ago for starlike functions in S [**Lö1**, **Ne**], and for functions in S with real coefficients [**Di**, **Rog**, **Sz**]. In 1955, Hayman [**Ha1**, **Ha2**] proved the asymptotic result that for each $f \in S$, $\lim_{n\to\infty} \frac{|a_n|}{n}$ exists and is less than 1 except for rotations of the Koebe function. Good historical articles are found in [BDDM] and [Du1]; much of the background mathematics is presented in [Du2].

This paper is organized as follows. In section 2, we present Bieberbach's proof of his conjecture for n=2, using the area theorem. (The equality part of the n=2 case is required in section 6.) In section 3, we present the conjectures of Robertson [Rob] and Milin [Mi3], and show using a Lebedev-Milin inequality [**LM**, **Mi2**, **Mi3**] that

$$\text{Milin Conj.} \implies \text{Robertson Conj.} \implies \text{Bieberbach Conj.}$$

In section 4, we discuss topological considerations, and reduce the problem to consideration of single-slit mappings. In section 5, we present de Branges's proof of the inequality of the Milin conjecture, following [FG]. The proof relies heavily upon Löwner's representation [Lö2], and uses some special functions introduced by de Branges in his proof, as well as an inequality of Askey and Gasper [AG]. In section 6 we show that equality holds only for rotations of the Koebe function.

2. The second coefficient. The area theorem.

Related to S is the class Σ of all univalent analytic functions g defined in the annulus $|z| > 1$, with Laurent expansion of the form

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + ..., \qquad |z| > 1.$$

The following important theorem was proved by T.H. Gronwall in 1914.

The Area Theorem. *If* $g(z) = z+b_0+b_1z^{-1}+b_2z^{-2}+... \in \Sigma$, *then* $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

Proof. For $r > 1$, let C_r be the image under g of the circle $|z| = r$. The C_r is a simple, closed curve since g is univalent. By Green's Theorem, the area of C_r is given by

$$\begin{aligned}
\text{area}(C_r) &= \frac{1}{2i}\int_{C_r} (ix\,dy - iy\,dx) \\
&= \frac{1}{2i}\left(\int_{C_r} (ix\,dy - iy\,dx) + \int_{C_r} (x\,dx + y\,dy)\right) \\
&= \frac{1}{2i}\int_{C_r} (x - iy)(dx + idy) \\
&= \frac{1}{2i}\int_{C_r} \overline{w}\,dw \\
&= \frac{1}{2i}\int_{|z|=r} \overline{g(z)}g'(z)\,dz \\
&= \frac{1}{2i}\int_{|z|=r} \left(\overline{z} + \overline{b_0} + \sum_{n=1}^{\infty} \overline{b_n z^{-n}}\right)\left(1 - \sum_{n=1}^{\infty} nb_n z^{-n-1}\right) dz \\
&= \frac{1}{2i}\int_0^{2\pi} \left(re^{-i\theta} + \overline{b}_0 + \sum_{n=1}^{\infty} \overline{b}_n r^{-n} e^{in\theta}\right)\left(1 - \sum_{n=1}^{\infty} nb_n r^{-n-1} e^{-i(n+1)\theta}\right) ire^{i\theta}\,d\theta \\
&= \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right).
\end{aligned}$$

But the area of C_r is non-negative, so we have

$$r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \geq 0, \qquad r > 1.$$

Now, if we had

$$\sum_{n=1}^{\infty} n|b_n|^2 > 1,$$

then we could find a positive integer N, and $\alpha > 0$, such that

$$\sum_{n=1}^{N} n|b_n|^2 = 1 + \alpha.$$

Choose $r > 1$ such that

$$r^{-2N} > \frac{1 + \alpha/2}{1 + \alpha},$$

and

$$r^2 < 1 + \alpha/4.$$

Then

$$\begin{aligned} r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} &\leq r^2 - \sum_{n=1}^{N} n|b_n|^2 r^{-2n} \\ &\leq r^2 - r^{-2N} \sum_{n=1}^{N} n|b_n|^2 \\ &< (1 + \alpha/4) - \frac{1 + \alpha/2}{1 + \alpha}(1 + \alpha) \\ &= -\alpha/4 \\ &< 0, \end{aligned}$$

contradicting the above result. □

Corollary. *$|b_1| \leq 1$, and $|b_1| = 1$ if and only if $g(z) = z + b_0 + \alpha/z$, where $|\alpha| = 1$.*

The above theorem allows us to prove Bieberbach's conjecture for the second coefficient, as proved by Bieberbach [Bi] in 1916.

Theorem (Bieberbach). *Let $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in S$. Then $|a_2| \leq 2$, and $|a_2| = 2$ only when f is a rotation of the Koebe function.*

Proof. Let g(z) be the unique odd function analytic in $|z| > 1$ such that

$$g(z) = z + b_1 z^{-1} + b_3 z^{-3} + ...,$$

and

$$g(z)^2 = \frac{1}{f(1/z^2)}, \qquad |z| > 1.$$

(Such a function exists since $f(1/z^2)$ is non-zero and even.) We can write

$$g(z) = \frac{1}{\sqrt{f(1/z^2)}}$$

provided we choose the appropriate branch of the square root at each z. Now, g is univalent since if $g(z_1) = g(z_2)$, then $f(1/z_1^2) = f(1/z_2^2)$, so since f is univalent, $z_1 = \pm z_2$. But then the oddness of g implies $z_1 = z_2$, since g is non-zero. Hence, $g \in \Sigma$, so by the Area Theorem, $|b_1| \leq 1$.

Now,

$$f(1/z^2) = z^{-2} + a_2 z^{-4} + ...,$$

so

$$\frac{1}{f(1/z^2)} = z^2 - a_2 + ...$$

But

$$g(z)^2 = z^2 + 2b_1 + \left(2b_3 + b_1^2\right) z^{-2} + ...,$$

so we must have $b_1 = -a_2/2$. Thus, $|b_1| \leq 1$ implies $|a_2| \leq 2$.

If $|a_2| = 2$, then $|b_1| = 1$, so by the above corollary

$$g(z) = z + \alpha/z, \qquad \text{for some } \alpha \in \mathbb{C} \text{ with } |\alpha| = 1.$$

Then

$$f(z) = \frac{1}{g(1/\sqrt{z})^2} = \frac{1}{(1/\sqrt{z} + \alpha\sqrt{z})^2} = \frac{z}{(1+\alpha z)^2} = -\alpha^{-1} k(-\alpha z),$$

a rotation of the Koebe function. □

3. The Robertson and Milin conjectures.

In 1936, M.S. Robertson [Rob] conjectured the following.

The Robertson Conjecture. *Let $p(z) = z + c_3 z^3 + c_5 z^5 + ... \in S$ be odd. Then (letting $c_1 = 1$), we have $|c_1|^2 + |c_3|^2 + ... + |c_{2n-1}|^2 \leq n$, for $n = 2, 3, 4, ...$ Furthermore, equality occurs for any one n only when p satisfies $p(z)^2 = r(z^2)$, where r is a rotation of the Koebe function.*

Theorem. *For each $n = 2, 3, 4, ...,$ the Robertson conjecture for n implies the Bieberbach conjecture for n.*

Proof. Assume that the Robertson conjecture holds for n, and let

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \in S.$$

Let $p(z)$ be the unique odd function analytic in $|z| < 1$ such that

$$p(z) = z + c_3 z^3 + c_5 z^5 + ...,$$

and such that

$$p(z)^2 = f(z^2).$$

We may write

$$p(z) = \sqrt{f(z^2)}$$

provided we choose the appropriate branch of the square root for each z. Now, p is univalent, for if $p(z_1) = p(z_2)$, then $f(z_1^2) = f(z_2^2)$, so since f is univalent, $z_1 = \pm z_2$. But then the oddness of p implies $z_1 = z_2$ (if $p(z_1) = p(z_2) = 0$, then $z_1 = z_2 = 0$ by the univalence of f). Hence, p is an odd function in S, and the Robertson conjecture applies.

Since

$$f(z^2) = p(z)^2,$$

comparing coefficients shows

$$a_n = c_1 c_{2n-1} + c_3 c_{2n-3} + ... + c_{2n-1} c_1,$$

so

$$\begin{aligned} |a_n| &\leq |c_1|\ |c_{2n-1}| + |c_3|\ |c_{2n-3}| + ... + |c_{2n-1}|\ |c_1|, \\ &= v \cdot w, \end{aligned}$$

where

$$v = (|c_1|, |c_3|, ..., |c_{2n-1}|)\,,$$

and

$$w = (|c_{2n-1}|, |c_{2n-3}|, ..., |c_1|)\,.$$

Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned} |a_n| &\leq \|v\|\ \|w\| \\ &= \|v\|^2 \\ &= |c_1|^2 + |c_3|^2 + ... + |c_{2n-1}|^2. \end{aligned}$$

The Robertson conjecture thus implies that $|a_n| \leq n$. Furthermore, if $|a_n| = n$, then $|c_1|^2 + |c_3|^2 + ... + |c_{2n-1}|^2 = n$, so the Robertson conjecture implies that $p(z)^2 = r(z^2)$, where r is a rotation of the Koebe function. But then $f(z^2) = r(z^2)$, so comparing power series shows $f(z) = r(z)$, and f is a rotation of the Koebe function. □

Given a function $f \in S$, we define its *logarithmic coefficients* $\{\gamma_n\}$ by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \qquad |z| < 1.$$

(Note that, letting $g(z) = \frac{f(z)}{z}$, we have

$$g(z) = 1 + a_2 z + a_3 z^2 + ...,$$

so $g(0) = 1 \neq 0$. Furthermore, $g(z)$ is not zero elsewhere in $|z| < 1$ by the univalence of f. Hence, we may formally define $\log \frac{f(z)}{z}$ as the integral from 0 to z of $g'(z)/g(z)$, which is single-valued. The integral is independent of path since the disk $|z| < 1$ is simply connected. Thus $\log \frac{f(z)}{z}$ is analytic in $|z| < 1$.) The logarithmic coefficients of the Koebe functions are easily computed. We have

$$\log \frac{k(z)}{z} = \log(1-z)^{-2} = -2\log(1-z) = 2\left(z + \frac{z^2}{2} + \frac{z^3}{3} + ...\right),$$

so the Koebe function satisfies $\gamma_n = 1/n$, for $n = 1, 2, 3, ...$

In 1971, Milin [Mi3] made the following conjecture.

The Milin Conjecture. *For each function $f \in S$, its logarithmic coefficients satisfy*

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0,$$

or equavilantly

$$\sum_{k=1}^{n}(n-k+1)\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0,$$

for $n = 1, 2, 3, \ldots$. Furthermore, we have equality for any one n only when f is a rotation of the Koebe function.

We wish to show that the Milin conjecture implies the Robertson (and hence the Bieberbach) conjecture. We require the following very general inequality from [**LM**, **Mi2**, **Mi3**]. Our proof follows [Du2, §5.1].

The Lebedev-Milin Exponentiation Inequality. *Let*

$$\phi(z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \ldots$$

be any complex power series with radius of convergence $R > 0$. Write

$$e^{\phi(z)} = 1 + \beta_1 z + \beta_2 z^2 + \ldots, \qquad |z| < R.$$

Then for $n = 1, 2, 3, \ldots$, we have (letting $\beta_0 = 1$)

$$\frac{1}{n+1}\sum_{k=0}^{n}|\beta_k|^2 \leq \exp\left(\frac{1}{n+1}\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right).$$

Proof. Letting $\psi(z) = e^{\phi(z)}$, we have $\psi'(z) = \phi'(z)\psi(z)$, i.e.

$$\sum_{k=1}^{\infty} k\beta_k z^{k-1} = \left(\sum_{k=1}^{\infty} k\alpha_k z^{k-1}\right)\left(\sum_{k=0}^{\infty}\beta_k z^k\right),$$

so comparing coefficients yields

$$k\beta_k = \sum_{j=0}^{k-1}\beta_j (k-j)\alpha_{k-j}, \qquad k = 1, 2, 3, \ldots$$

Hence

$$\begin{aligned}|k\beta_k| &\leq \sum_{j=0}^{k-1}|(k-j)\alpha_{k-j}|\ |\beta_j| \\ &= v \cdot w,\end{aligned}$$

where

$$v = (k|\alpha_k|, (k-1)|\alpha_{k-1}|, \ldots, |\alpha_1|),$$

and

$$w = (|\beta_0|, |\beta_1|, ..., |\beta_{k-1}|).$$

Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned} k^2|\beta_k|^2 &\leq \|v\|^2 \, \|w\|^2 \\ &= \left(\sum_{j=1}^{k} j^2|\alpha_j|^2\right) \left(\sum_{j=0}^{k-1} |\beta_j|^2\right). \end{aligned}$$

Let

$$A_k = \sum_{j=1}^{k} j^2|\alpha_j|^2, \qquad \text{and } B_k = \sum_{j=0}^{k} |\beta_k|^2.$$

Then the above equation becomes

$$k^2|\beta_k|^2 \leq A_k B_{k-1}.$$

Hence,

$$\begin{aligned} B_n &= B_{n-1} + |\beta_n|^2 \\ &\leq B_{n-1} + \frac{1}{n^2} A_n B_{n-1} \\ &= B_{n-1}\left(1 + \frac{A_n}{n^2}\right) \\ &= B_{n-1}\left(\frac{n+1}{n}\right)\left(\frac{n}{n+1} + \frac{A_n}{n(n+1)}\right) \\ &= B_{n-1}\left(\frac{n+1}{n}\right)\left(1 + \frac{A_n - n}{n(n+1)}\right) \\ &\leq B_{n-1}\left(\frac{n+1}{n}\right)\exp\left(\frac{A_n - n}{n(n+1)}\right). \end{aligned}$$

But $B_0 = |\beta_0|^2 = 1$, so using induction, we have

$$\begin{aligned} B_n &\leq \prod_{k=1}^{n}\left(\frac{k+1}{k}\right)\exp\left(\frac{A_k - k}{k(k+1)}\right) \\ &= (n+1)\exp\left(\sum_{k=1}^{n}\left(\frac{A_k - k}{k(k+1)}\right)\right) \\ &= (n+1)\exp\left(\sum_{k=1}^{n}\frac{A_k}{k(k+1)} - \sum_{k=1}^{n}\frac{1}{k+1}\right). \end{aligned}$$

Now,

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{A_k}{k(k+1)} &= \sum_{k=1}^{n} A_k \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \sum_{j=1}^{k} j^2 |\alpha_j|^2 \\
&= \sum_{j=1}^{n} j^2 |\alpha_j|^2 \sum_{k=j}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \sum_{j=1}^{n} j^2 |\alpha_j|^2 \left(\frac{1}{j} - \frac{1}{n+1}\right) \\
&= \sum_{j=1}^{n} j \left(1 - \frac{j}{n+1}\right) |\alpha_j|^2 \\
&= \frac{1}{n+1} \sum_{j=1}^{n} (n+1-j) j |\alpha_j|^2.
\end{aligned}
$$

Also

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k+1} &= \sum_{k=2}^{n+1} \frac{1}{k} \\
&= \sum_{k=1}^{n} \frac{1}{k} - 1 + \frac{1}{n+1} \\
&= \sum_{k=1}^{n} \frac{1}{k} - \frac{n}{n+1} \\
&= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{n+1} \\
&= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{n+1}\right) \\
&= \frac{1}{n+1} \sum_{k=1}^{n} \frac{n+1-k}{k}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
B_n &\le (n+1) \exp\left(\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k) k |\alpha_k|^2 - \frac{1}{n+1} \sum_{k=1}^{n} \left(\frac{n+1-k}{k}\right)\right) \\
&= (n+1) \exp\left(\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k) \left(k |\alpha_k|^2 - \frac{1}{k}\right)\right) \\
&= (n+1) \exp\left(\frac{1}{n+1} \sum_{m=1}^{n} \sum_{k=1}^{m} \left(k |\alpha_k|^2 - \frac{1}{k}\right)\right),
\end{aligned}
$$

which gives the result. □

Theorem. *For each $n = 1, 2, 3, ...$, the Milin conjecture for n implies the Robertson conjecture for $n + 1$.*

Proof. Assume the Milin conjecture holds for n, and let $h \in S$ be odd. Then $h(z)^2$ is even, so

$$h(z)^2 = f(z^2)$$

for some function f analytic in $|z| < 1$. Furthermore, f is univalent, for if $f(z_1) = f(z_2)$, then choosing ζ_1, ζ_2 with $\zeta_1^2 = z_1$ and $\zeta_2^2 = z_2$, we have $f(\zeta_1^2) = f(\zeta_2^2)$, so $h(\zeta_1)^2 = h(\zeta_2)^2$. Then the univalence and oddness of h implies $\zeta_1 = \pm\zeta_2$, so $z_1 = z_2$. Hence, $f \in S$. Let $\{\gamma_n\}$ be its logarithmic coefficients, so that

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

Write

$$h(z) = z + c_3 z^3 + c_5 z^5 + ...$$

Then

$$\begin{aligned} \frac{h(\sqrt{z})}{\sqrt{z}} &= 1 + c_3 z + c_5 z^2 + ... \\ &= \sum_{n=0}^{\infty} c_{2n+1} z^n \qquad (\text{where } c_1 = 1). \end{aligned}$$

Now,

$$\begin{aligned} \log\left(\frac{h(\sqrt{z})}{\sqrt{z}}\right) &= \log\left(\sqrt{\frac{f(z)}{z}}\right) \\ &= \frac{1}{2} \log \frac{f(z)}{z} \\ &= \sum_{n=1}^{\infty} \gamma_n z^n. \end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} c_{2n+1} z^n = \frac{h(\sqrt{z})}{\sqrt{z}} = \exp\left(\sum_{n=1}^{\infty} \gamma_n z^n\right).$$

The Lebedev-Milin Exponentiation Inequality then says that

$$\frac{1}{n+1} \sum_{k=0}^{n} |c_{2k+1}|^2 \leq \exp\left(\frac{1}{n+1} \sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k}\right)\right),$$

and Milin's conjecture then implies that

$$\frac{1}{n+1} \sum_{k=0}^{n} |c_{2k+1}|^2 \leq 1,$$

which is the inequality of Robertson's conjecture for $n+1$. Furthermore, if

$$\frac{1}{n+1}\sum_{k=0}^{n}|c_{2k+1}|^2 = 1,$$

then we must have

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) = 0.$$

The Milin conjecture then implies that f is a rotation of the Koebe function, so that

$$h(z)^2 = f(z^2),$$

with f a rotation of the Koebe function as required. $\square$

4. Topological considerations. Single-slit mappings.

We endow S with the topology Δ induced by *locally uniform convergence*, i.e. uniform convergence on every compact subset of the unit disk. This topology can be metrized (see [Po, p. 27-28]) by

$$d(f,g) = \sum_{k=2}^{\infty} 2^{-k}\arctan\left(\sup_{z\in A_k}|f(z)-g(z)|\right),$$

where

$$A_k = \left\{z\in\mathbb{C}\ \middle|\ |z|\le 1-\frac{1}{k}\right\}.$$

Lemma 4-1. *The mapping $f \mapsto \gamma_k$ from S to $\mathbb{C}$ taking each $f\in S$ to its k^{th} logarithmic coefficient is continuous in the topology Δ.*

Proof. Let $\{f_n\}\subseteq S$ be a sequence of functions converging to f in the topology Δ. Write

$$\log\frac{f_n(z)}{z} = 2\sum_{j=1}^{\infty}\gamma_{n,j}z^j$$

and

$$\log\frac{f(z)}{z} = 2\sum_{j=1}^{\infty}\gamma_j z^j.$$

We wish to show $\{\gamma_{n,k}\}\to\gamma_k$. Let

$$K = \left\{z\in\mathbb{C}\ \middle|\ |z| = 1/2\right\}.$$

Then

$$\left\{\sup_{z\in K}|f(z)-f_n(z)|\right\}\to 0,$$

so

$$\left\{\sup_{z\in K}\left|\log\frac{f(z)}{z} - \log\frac{f_n(z)}{z}\right|\right\}\to 0,$$

provided we choose the branch of log in each case so that $z = 0$ gets mapped to 0. But then the Cauchy estimate (at $r = 1/2$) for the k^{th} coefficient of the power series expansion around zero of

$$\log \frac{f(z)}{z} - \log \frac{f_n(z)}{z}$$

shows that

$$\{|\gamma_{n,k} - \gamma_k|\} \to 0,$$

as desired. □

We define a *single-slit mapping* to be a function analytic in the unit disk, such that its range is equal to the entire complex plane except for a single Jordan arc extending from a finite point to infinity. We let S' be the set of all single-slit mappings in S.

Theorem 4-2. *S' is dense in S with respect to the topology Δ.*

Proof. Let $f \in S$. For each $r \in \mathbb{R}$, $0 < r < 1$, the function $f_r(z) = r^{-1} f(rz)$ is in S. Let $\{r_n\}$ be a sequence of positive real numbers increasing to 1. I claim that

$$\{f_{r_n}\} \to f$$

in the topology Δ. Indeed, given any compact subset K of the unit disk, let $R = \sup_{z \in K} |z|$, and let

$$T = \sup_{|z| \leq R} |f'(z)|,$$

and

$$U = \sup_{|z| \leq R} |f(z)|.$$

Then

$$\begin{aligned}
\sup_{z \in K} |f_{r_n}(z) - f(z)| &= \sup_{z \in K} |r_n^{-1} f(r_n z) - f(z)| \\
&\leq \sup_{z \in K} |r_n^{-1} f(r_n z) - f(r_n z)| + \sup_{z \in K} |f(r_n z) - f(z)| \\
&\leq (r_n^{-1} - 1)U + (1 - r_n)T \\
&\to 0 \text{ as } r_n \to 1^-.
\end{aligned}$$

Hence, it suffices to show we can approximate f_r arbitrarily closely (with respect to Δ) by functions in S', for each $0 < r < 1$. To do this, it suffices to find, for each $0 < r < 1$, a sequence $\{g_n\}$ of functions in S' which converges to f_r with respect to Δ.

Let $0 < r < 1$. Let

$$J = \left\{ f_r(z) \;\middle|\; |z| = 1 \right\} = \left\{ r^{-1} f(rz) \;\middle|\; |z| = 1 \right\}.$$

Then J is a simple, closed curve by the univalence of f. Choose $w \in J$. Let $\{\Gamma_n\}$ be an increasing sequence of Jordan arcs contained in J each of which begins at w and proceeds counter-clockwise, and such that

$$\bigcup_{n=1}^{\infty} \Gamma_n = J.$$

Let Γ be a Jordan arc from w to infinity not touching J except at w. Then $\Gamma \cup \Gamma_n$ is a Jordan arc from a finite point to infinity. For each n, let D_n be the complement of $\Gamma \cup \Gamma_n$, and let

$$D = \left\{ f_r(z) \;\middle|\; |z| < 1 \right\} = \left\{ r^{-1} f(rz) \;\middle|\; |z| < 1 \right\}.$$

The D is the *kernel* of (i.e. the largest open connected set contained in the intersection of) every subsequence of $\{D_n\}$. By the Riemann Mapping Theorem, for each n we can find a univalent (and single-slit) mapping h_n from the unit disk onto D_n, such that $h_n(0) = 0$ and $h_n'(0) > 0$. The Carathéodory (Kernel) Convergence Theorem (see [Du2, Theorem 3.1] or [Po, Theorem 1.8]) then implies that $\{h_n\}$ converges in the topology Δ to some univalent function η from the unit disk onto D. But then we must have $\eta(0) = 0$ and $\eta'(0) > 0$, so the Riemann Mapping Theorem implies that $\eta = f_r$, i.e. $\{h_n\} \to f_r$ in the topology Δ. Hence $\{h_n'(0)\}$ converges to 1, so for sufficiently large n we can let

$$g_n(z) = \frac{h_n(z)}{h_n'(0)}$$

to get $g_n \in S'$, and $\{g_n\} \to f_r$ in the topology Δ. □

Theorem 4-3. *It suffices to prove the inequality of the Milin conjecture for functions in S'.*

Proof. Given $f \in S$, choose $\{f_n\} \subseteq S'$ with $\{f_n\} \to f$ in the topology Δ. Write

$$\log \frac{f_n(z)}{z} = 2 \sum_{j=1}^{\infty} \gamma_{n,j} z^j$$

and

$$\log \frac{f(z)}{z} = 2 \sum_{j=1}^{\infty} \gamma_j z^j.$$

By Lemma 4-1, $\{\gamma_{n,k}\} \to \gamma_k$ for each k. Hence, if the inequality of the Milin conjecture holds for each f_n, then it also holds for f. □

We shall require the following standard theorem (or, rather, its corollary) for a technical reason in the next section. We do not prove it here; see [Po, Theorem 1.7] or [Du2, p. 9].

Theorem 4-4. *S is compact in the topology Δ.*

From the theorem, we easily obtain

Corollary 4-5. *For each $k = 1, 2, 3, \ldots$, the supremum*

$$M_k = \sup_{f \in S} |\gamma_k|$$

is finite.

Proof. From Lemma 4-1 and Theorem 4-4, it follows that $|\gamma_k|$ attains its maximum in S. □

5. de Branges's proof of the inequality of the Milin conjecture.

We choose a function $f \in S'$ (see Theorem 4-3), and a positive integer n. We let $\{\gamma_k\}$ be the logarithmic coefficients of f:

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k, \qquad |z| < 1.$$

We wish to show that

$$\sum_{k=1}^{n} (n-k+1) \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0. \tag{5-1}$$

The plan of the proof is as follows. We shall define a differentiable function

$$\phi : [0, \infty) \to \mathbb{R}.$$

We shall show that

$$\phi(0) = 4 \sum_{k=1}^{n} (n-k+1) \left(k|\gamma_k|^2 - \frac{1}{k} \right),$$

and that

$$\lim_{t \to \infty} \phi(t) = 0.$$

We shall then show that

$$\phi'(t) \geq 0, \qquad t \in (0, \infty),$$

to conclude that $\phi(0) \leq 0$, proving (5-1).

We require the following fundamental result of Löwner [Lö2], which we shall not prove. See [Du2, Theorem 3.3] for a similar result, also from [Lö2], from which the stated result can easily be derived.

The Löwner Representation Theorem. *Let $f \in S'$. Then there is a parameterized family of univalent functions*

$$g(z,t) = e^t z + a_2(t) z^2 + a_3(t) z^3 + ..., \qquad |z| < 1, \quad t \in [0, \infty)$$

such that

$$g(z, 0) = f(z),$$

and

$$\dot{g}(z,t) = \frac{1 + \kappa(t) z}{1 - \kappa(t) z} \, z \, g'(z,t), \qquad |z| < 1, t \in [0, \infty), \tag{5-2}$$

for some continuous function

$$\kappa : [0, \infty) \to \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\},$$

where

$$\dot{g} \equiv \frac{\partial g}{\partial t} \qquad \textit{and} \qquad g' \equiv \frac{\partial g}{\partial z}.$$

The differential equation (5-2) is called the *Löwner differential equation.*

Choose a parameterized family $g(z,t)$ for f from the Löwner Representation Theorem. Note that for each t, the function $z \mapsto e^{-t}g(z,t)$ is in S. Define $\{c_k(t)\}$ by

$$\log \frac{e^{-t}g(z,t)}{z} = \sum_{k=1}^{\infty} c_k(t) z^k, \qquad |z| < 1.$$

Then since $g(z,0) = f(z)$, we have that

(5-3) $$c_k(0) = 2\gamma_k, \qquad k = 1, 2, 3, ...$$

For $k = 1, 2, ..., n$, let

$$\tau_k(t) = k \sum_{j=0}^{n-k} (-1)^j \frac{(2k+j+1)_j (2k+2j+2)_{n-k-j}}{(k+j)j!(n-k-j)!} e^{-(j+k)t},$$

where for $a \in \mathbb{R}$ we define $(a)_0 = 1$ and

$$(a)_s = a(a+1)(a+2)...(a+s-1), \qquad s \geq 1.$$

Let

$$\phi(t) = \sum_{k=1}^{n} \left(k|c_k(t)|^2 - \frac{4}{k} \right) \tau_k(t), \qquad t \in [0, \infty).$$

Theorem 5-1. $\lim_{t\to 0} \phi(t) = 0$.

Proof. By Corollary 4-5, since the functions $z \mapsto e^t g(z,t)$ are in S, $|c_k(t)|$ are bounded as functions of t. Also, directly from the definition of $\tau(t)$, we have $\lim_{t\to 0} \tau(t) = 0$. The result follows. □

Lemma 5-2. *We have*

(5-4) $$\tau_k(t) - \tau_{k+1}(t) = -\frac{\tau_k'(t)}{k} - \frac{\tau_{k+1}'(t)}{k+1}, \qquad k = 1, 2, ..., n-1,$$

and

$$\tau_n(t) = -\frac{\tau_n'(t)}{n}.$$

Proof. We have that

$$\tau_k(t) = k \sum_{j=0}^{n-k} (-1)^j \frac{(2k+j+1)_j (2k+2j+2)_{n-k-j}}{(k+j)j!(n-k-j)!} e^{-(j+k)t},$$

so

$$\begin{aligned}\tau_{k+1}(t) &= (k+1) \sum_{j=0}^{n-k-1} (-1)^j \frac{(2k+j+3)_j (2k+2j+4)_{n-k-1-j}}{(k+1+j)j!(n-k-1-j)!} e^{-(j+k+1)t} \\ &= (k+1) \sum_{j=1}^{n-k} (-1)^{j-1} \frac{(2k+j+2)_{j-1} (2k+2j+2)_{n-k-j}}{(k+j)(j-1)!(n-k-j)!} e^{-(j+k)t}.\end{aligned}$$

Let

$$A_{k,j} = (-1)^j \frac{(2k+2j+2)_{n-k-j}}{(k+j)j!(n-k-j)!}.$$

Then

$$\tau_k(t) = k\sum_{j=0}^{n-k}(2k+j+1)_j A_{k,j}e^{-(j+k)t} \tag{5-5}$$

and

$$\tau_{k+1}(t) = -(k+1)\sum_{j=1}^{n-k} j(2k+j+2)_{j-1}A_{k,j}e^{-(j+k)t}, \tag{5-6}$$

so

$$\tau_k(t)-\tau_{k+1}(t) = kA_{k,0}e^{-kt}+\sum_{j=1}^{n-k}\big(k(2k+j+1)_j+(k+1)j(2k+j+2)_{j-1}\big)A_{k,j}e^{-(j+k)t}.$$

Differentiating (5-5) and (5-6) yields

$$\tau_k'(t) = -k\sum_{j=0}^{n-k}(j+k)(2k+j+1)_j A_{k,j}e^{-(j+k)t}$$

and

$$\tau_{k+1}'(t) = +(k+1)\sum_{j=1}^{n-k}(j+k)j(2k+j+2)_{j-1}A_{k,j}e^{-(j+k)t},$$

so

$$-\frac{\tau_k'(t)}{k}-\frac{\tau_{k+1}'(t)}{k+1} = kA_{k,0}e^{-kt}+\sum_{j=1}^{n-k}\big((j+k)(2k+j+1)_j-(j+k)j(2k+j+2)_{j-1}\big)A_{k,j}e^{-(j+k)t}.$$

Hence, equation (5-4) will follow from showing that

$$(j+k)(2k+j+1)_j-(j+k)j(2k+j+2)_{j-1} = k(2k+j+1)_j+(k+1)j(2k+j+2)_{j-1},$$

which is the same thing as

$$j(2k+j+1)_j = j(2k+j+1)(2k+j+2)_{j-1},$$

a trivial identity. Lastly, from (5-5), $\tau_n(t) = nA_{n,0}e^{-nt}$, so $\tau_n'(t) = -n\tau_n(t)$, proving the second statement. □

We let $P_j^{(\alpha,\beta)}(x)$ be the *Jacobi polynomials*, defined by

$$P_j^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_j}{j!}\sum_{s=0}^{j}\frac{(-j)_s(j+\alpha+\beta+1)_s}{s!(\alpha+1)_s}\left(\frac{1-x}{2}\right)^s. \tag{5-7}$$

Recall that

$$P_j^{(\alpha,\beta)}(x) = (-1)^j P_j^{(\beta,\alpha)}(-x),$$

and that

$$P_j^{(\alpha,\beta)}(1) = \binom{j+\alpha}{j}.$$

It follows immediately that

$$P_j^{(\alpha,0)}(-1) = (-1)^j.$$

Lemma 5-3. *We have*

$$\tau_k'(t) = -ke^{-kt}\sum_{j=0}^{n-k} P_j^{(2k,0)}\left(1-2e^{-t}\right), \qquad k=1,2,...,n.$$

Proof. We have that

$$\begin{aligned}\sum_{j=0}^{n-k} P_j^{(2k,0)}(1-2e^{-t}) &= \sum_{j=0}^{n-k}\frac{(2k+1)_j}{j!}\sum_{s=0}^{j}\frac{(-j)_s(j+2k+1)_s}{s!(2k+1)_s}\left(e^{-t}\right)^s\\ &= \sum_{s=0}^{n-k}\sum_{j=s}^{n-k}\frac{(2k+1)_j}{j!}\frac{(-j)_s(j+2k+1)_s}{s!(2k+1)_s}\left(e^{-t}\right)^s\\ &= \sum_{s=0}^{n-k}\sum_{j=0}^{n-k-s}\frac{(2k+1)_{j+s}}{(j+s)!}\frac{\big(-(j+s)\big)_s(j+s+2k+1)_s}{s!(2k+1)_s}e^{-st}\\ &= \sum_{s=0}^{n-k}\frac{(2k+1)_{2s}}{s!(2k+1)_s}(-1)^s e^{-st}\sum_{j=0}^{n-k-s}\frac{(2s+2k+1)_j}{j!},\end{aligned}$$

the last equality following from the fact that

$$\begin{aligned}&\frac{(2k+1)_{j+s}\big(\text{-}(j+s)\big)_s(j+s+2k+1)}{(j+s)!}\\ &= \frac{(2k+1)...(2k+j+s)\big(\text{-}(j+s)\big)...\big(\text{-}(j+1)\big)(j+s+2k+1)...(j+2s+2k)}{(j+s)!}\\ &= \frac{(-1)^s(j+1)_s(2k+1)_{2s+j}}{(j+s)!}\\ &= \frac{(-1)^s(2k+1)_{2s+j}}{j!}\\ &= \frac{(-1)^s(2k+1)...(2k+2s)(2k+2s+1)...(2k+2s+j)}{j!}\\ &= \frac{(-1)^s(2k+1)_{2s}(2s+2k+1)_j}{j!}.\end{aligned}$$

Hence,

$$\text{(5-8)}\qquad \sum_{j=0}^{n-k} P_j^{(2k,0)}\left(1-2e^{-t}\right) = \sum_{s=0}^{n-k}\frac{(2k+1)_{2s}}{s!(2k+1)_s}(-1)^s e^{-st}\sum_{j=0}^{n-k-s}\frac{(2s+2k+1)_j}{j!}.$$

Now, from the identity

$$\binom{p}{q}+\binom{p}{q+1}=\binom{p+1}{q+1}$$

we obtain

$$\sum_{j=0}^{N}\binom{a+j-1}{j}=\binom{a+N}{N}, \qquad a\in\mathbb{N},$$

i.e.

$$\sum_{j=0}^{N} \frac{(a)_j}{j!} = \frac{(a+1)_N}{N!}, \qquad a \in \mathbb{N}.$$

(In fact, since this last equation involves polynomials in a, it is valid for all real a, but we don't need that fact here.) Using this in (5-8) with $N = n-k-s$ and $a = 2s+2k+1$ yields

$$\begin{aligned}\sum_{j=0}^{n-k} P_j^{(2k,0)}\left(1-2e^{-t}\right) &= \sum_{s=0}^{n-k} \frac{(2k+1)_{2s}}{s!(2k+1)_s}(-1)^s e^{-st}\frac{(2s+2k+2)_{n-k-s}}{(n-k-s)!} \\ &= \sum_{s=0}^{n-k} \frac{(2k+s+1)_s}{s!}(-1)^s e^{-st}\frac{(2s+2k+2)_{n-k-s}}{(n-k-s)!}.\end{aligned}$$

Hence,

$$-ke^{-kt}\sum_{j=0}^{n-k} P_j^{(2k,0)}\left(1-2e^{-t}\right) = -k\sum_{s=0}^{n-k}(-1)^s \frac{(2k+s+1)_s(2k+2s+2)_{n-k-s}}{s!(n-k-s)!}e^{-(k+s)t},$$

and this expression is equal to $\tau_k'(t)$ directly from the definition of $\tau_k(t)$. □

Theorem 5-4. *We have*

$$\phi(0) = 4\sum_{k=0}^{n}(n-k+1)\left(k|\gamma_k|^2 - \frac{1}{k}\right).$$

Proof. Using equation (5-3), we have that

$$\begin{aligned}\phi(0) &= \sum_{k=1}^{n}\left(k|c_k(0)|^2 - \frac{4}{k}\right)\tau_k(0) \\ &= 4\sum_{k=1}^{n}\left(k|\gamma_k|^2 - \frac{1}{k}\right)\tau_k(0),\end{aligned}$$

so it suffices to show that $\tau_k(0) = n-k+1$, for $k = 1, 2, ..., n$. By definition, $\tau_n(0) = n/n = 1$. By Lemma 5-3,

$$\begin{aligned}\frac{\tau_k'(0)}{-k} &= \sum_{j=0}^{n-k} P_j^{(2k,0)}(-1) \\ &= \sum_{j=0}^{n-k}(-1)^j \\ &= \begin{cases} 1, n-k \text{ even} \\ 0, n-k \text{ odd}\end{cases}\end{aligned}$$

Lemma 5-2 then implies that $\tau_k(0) - \tau_{k+1}(0) = 1$, for $k = 1, 2, ..., n-1$, and the result now follows by "descending induction". □

We require the following deep inequality of R. Askey and G. Gasper [AG, Theorem 3], which we shall not prove.

The Askey-Gasper Inequality. *If $\alpha > -2$, and N is any non-negative integer, then*

$$\sum_{j=0}^{N} P_j^{(\alpha,0)}(x) > 0, \qquad -1 < x \le 1.$$

Theorem 5-5. *We have*

$$\tau_k'(t) < 0, \qquad \text{for all } t \in (0,\infty), \qquad k = 1,2,...,n.$$

Proof. This is immediate from Lemma 5-3 and the Askey-Gasper Inequality, since $t \in (0,\infty)$ implies that $-1 < 1 - 2e^{-t} < 1$. □

Lemma 5-6. *For $k = 1,2,...,n$, $c_k(t)$ is differentiable, and*

$$c_k'(t) = 2\sum_{j=1}^{k-1} jc_j(t)\kappa(t)^{k-j} + kc_k(t) + 2\kappa(t)^k.$$

Proof. We have that

$$\log\frac{e^{-t}g(z,t)}{z} = \sum_{k=1}^{\infty} c_k(t)z^k, \qquad |z| < 1. \tag{5-9}$$

By equation (5-2), the function

$$z \mapsto \frac{\partial}{\partial t}\log\frac{e^{-t}g(z,t)}{z}$$

is analytic for $|z| < 1$, so we can write

$$\frac{\partial}{\partial t}\left(\sum_{k=1}^{\infty} c_k(t)z^k\right) = \sum_{k=0}^{\infty} d_k(t)z^k,$$

for some functions $d_k(t)$. Comparing coefficients in z^k then shows that $d_0(t) = 0$, and that each $c_k(t)$ is differentiable, with $c_k'(t) = d_k(t)$. In other words, we can differentiate equation (5-9) term-by-term with respect to t:

$$\begin{aligned}\sum_{k=1}^{\infty} c_k'(t)z^k &= \frac{\partial}{\partial t}\log\frac{e^{-t}g(z,t)}{z}\\ &= \frac{1}{\frac{e^{-t}g(z,t)}{z}}\left(\frac{-e^{-t}g(z,t)}{z} + \frac{e^{-t}\dot{g}(z,t)}{z}\right)\\ &= -1 + \frac{\dot{g}(z,t)}{g(z,t)},\end{aligned}$$

so

$$\frac{\dot{g}(z,t)}{g(z,t)} = 1 + \sum_{k=1}^{\infty} c_k'(t)z^k. \tag{5-10}$$

Differentiating (5-9) with respect to z yields

$$\sum_{k=1}^{\infty} kc_k(t)z^{k-1} = \frac{1}{\frac{e^{-t}g(z,t)}{z}}\left(\frac{e^{-t}g'(z,t)}{z} - \frac{e^{-t}g(z,t)}{z^2}\right)$$
$$= \frac{g'(z,t)}{g(z,t)} - \frac{1}{z},$$

so

$$z\frac{g'(z,t)}{g(z,t)} = 1 + \sum_{k=1}^{\infty} kc_k(t)z^k.$$

Combining this with (5-10) and using (5-3), we get

$$1 + \sum_{k=1}^{\infty} c_k'(t)z^k = \frac{\dot{g}(z,t)}{g(z,t)}$$
$$= \frac{1+\kappa(t)z}{1-\kappa(t)z} z\frac{g'(z,t)}{g(z,t)}$$
$$= \left(\frac{1+\kappa(t)z}{1-\kappa(t)z}\right)\left(1 + \sum_{k=1}^{\infty} kc_k(t)z^k\right)$$
$$= (1+\kappa(t)z)\left(1+\kappa(t)z+\kappa(t)z^2+...\right)\left(1 + \sum_{k=1}^{\infty} kc_k(t)z^k\right)$$
$$= \left(1 + 2\sum_{k=1}^{\infty}(\kappa(t)z)^k\right)\left(1 + \sum_{k=1}^{\infty} kc_k(t)z^k\right).$$

Comparing power series coefficients gives the result. $\square$

Lemma 5-7. *For $t \in [0,\infty)$, let $b_0(t) = 0$, and let*

$$b_k(t) = \sum_{j=1}^{k} jc_j(t)\kappa(t)^{-j}, \qquad k = 1, 2, ..., n.$$

Then

$$\phi'(t) = -\sum_{k=1}^{n} \mid b_{k-1}(t) + b_k(t) + 2 \mid^2 \frac{\tau_k'(t)}{k}.$$

Proof. We have

$$\phi(t) = \sum_{k=1}^{n}\left(kc_k(t)\overline{c_k(t)} - \frac{4}{k}\right)\tau_k(t),$$

so

$$\phi'(t) = \sum_{k=1}^{n}\left[k\left(c_k'(t)\overline{c_k(t)} + \overline{c_k'(t)}c_k(t)\right)\tau_k(t) + \left(kc_k(t)\overline{c_k(t)} - \frac{4}{k}\right)\tau_k'(t)\right]$$

(5-11)

$$= \sum_{k=1}^{n}\left[k\left(2 \text{ Re } c_k'(t)\overline{c_k(t)}\right)\tau_k(t) + \left(k|c_k(t)|^2 - \frac{4}{k}\right)\tau_k'(t)\right]$$

Now, note that

$$\begin{aligned}\kappa(t)^k\left(b_k(t)+b_{k-1}(t)+2\right) &= \kappa(t)^k\left(\sum_{j=1}^{k} jc_j(t)\kappa(t)^{-j}+\sum_{j=1}^{k-1} jc_j(t)\kappa(t)^{-j}+2\right)\\ &= \kappa(t)^k\left(2\sum_{j=1}^{k-1} jc_j(t)\kappa(t)^{-j}+kc_k(t)\kappa(t)^{-k}+2\right)\\ &= c_k'(t), \qquad \text{by Lemma 5-6.}\end{aligned}$$

Also

$$\left(\overline{b_k(t)}-\overline{b_{k-1}(t)}\right)\kappa(t)^{-k}=\left(k\overline{c_k(t)\kappa(t)^{+k}}\right)\kappa(t)^{-k}=k\overline{c_k(t)}.$$

Hence (5-11) becomes

$$\text{(5-12)}\quad \phi'(t)=\sum_{k=1}^{n} 2\ \mathrm{Re}\left[\left(\overline{b_k(t)}-\overline{b_{k-1}(t)}\right)\left(b_k(t)+b_{k-1}(t)+2\right)\right]\tau_k(t) + \sum_{k=1}^{n}\left(|kc_k(t)|^2-4\right)\frac{\tau_k'(t)}{k}.$$

Now,

$$\begin{aligned}&\mathrm{Re}\ \left[\left(\overline{b_k(t)}-\overline{b_{k-1}(t)}\right)\left(b_k(t)+b_{k-1}(t)+2\right)\right]\\ &\quad= \ \mathrm{Re}\ \left(|b_k(t)|^2+b_{k-1}(t)\overline{b_k(t)}-\overline{b_{k-1}(t)}b_k(t)-|b_{k-1}(t)|^2+2\overline{b_k(t)}-2\overline{b_{k-1}(t)}\right)\\ &\quad= |b_k(t)|^2-|b_{k-1}(t)|^2+2\ \mathrm{Re}\ b_k(t)-2\ \mathrm{Re}\ b_{k-1}(t).\end{aligned}$$

Therefore, the first sum in (5-12) may be written as

$$\begin{aligned}&\sum_{k=1}^{n}\left[2\left(|b_k(t)|^2-|b_{k-1}(t)|^2\right)+4\left(\ \mathrm{Re}\ b_k(t)-\ \mathrm{Re}\ b_{k-1}(t)\right)\right]\tau_k(t)\\ &\quad=\sum_{k=1}^{n}\left(2|b_k(t)|^2+4\ \mathrm{Re}\ b_k(t)\right)\tau_k(t)-\sum_{k=1}^{n-1}\left(2|b_k(t)|^2+4\ \mathrm{Re}\ b_k(t)\right)\tau_{k+1}(t)\\ &\quad=\left(2|b_n(t)|^2+4\ \mathrm{Re}\ b_n(t)\right)\tau_n(t)+\sum_{k=1}^{n-1}\left(2|b_k(t)|^2+4\ \mathrm{Re}\ b_k(t)\right)\left(\tau_k(t)-\tau_{k+1}(t)\right)\\ &\quad=\frac{-\tau_n'(t)}{n}\left(2|b_n(t)|^2+4\ \mathrm{Re}\ b_n(t)\right)-\sum_{k=1}^{n-1}\left(2|b_k(t)|^2+4\ \mathrm{Re}\ b_k(t)\right)\left(\frac{\tau_k'(t)}{k}+\frac{\tau_{k+1}'(t)}{k+1}\right)\\ &\quad=-\frac{\tau_n'(t)}{n}\left(2|b_n(t)|^2+4\ \mathrm{Re}\ b_n(t)\right)-\sum_{k=1}^{n-1}\left(2|b_k(t)|^2+4\ \mathrm{Re}\ b_k(t)\right)\frac{\tau_k'(t)}{k}\\ &\qquad\qquad-\sum_{k=2}^{n}\left(2|b_{k-1}(t)|^2+4\ \mathrm{Re}\ b_{k-1}(t)\right)\frac{\tau_k'(t)}{k}\\ &\quad=-\sum_{k=1}^{n}\left(2|b_k(t)|^2+4\ \mathrm{Re}\ b_k(t)+2|b_{k-1}(t)|^2+4\ \mathrm{Re}\ b_{k-1}(t)\right)\frac{\tau_k'(t)}{k}\end{aligned}$$

Noting also that $|kc_k(t)| = |b_k(t) - b_{k-1}(t)|$, (5-12) becomes

$$\phi'(t) = -\sum_{k=1}^{n}\left(2|b_k(t)|^2 + 4\ \mathrm{Re}\ b_k(t) + 2|b_{k-1}(t)|^2 + 4\ \mathrm{Re}\ b_{k-1}(t) - |b_k(t) - b_{k-1}(t)|^2 + 4\right)\frac{\tau_k'(t)}{k}$$

$$= -\sum_{k=1}^{n}\Big(2|b_k(t)|^2 + 2b_k(t) + 2\overline{b_k(t)} + 2|b_{k-1}(t)|^2 + 2b_{k-1}(t) + 2\overline{b_{k-1}(t)} - |b_k(t)|^2$$

$$- |b_{k-1}(t)|^2 + b_k(t)\overline{b_{k-1}(t)} + b_{k-1}(t)\overline{b_k(t)} + 4\Big)\frac{\tau_k'(t)}{k}$$

$$= -\sum_{k=1}^{n}\left|\ b_{k-1}(t) + b_k(t) + 2\ \right|^2 \frac{\tau_k'(t)}{k}. \quad \square$$

Theorem 5-8. *We have $\phi'(t) \geq 0$, for all $t \in (0, \infty)$.*

Proof. This is immediate from Theorem 5-5 and Lemma 5-7. $\square$

From Theorems 5-1 and 5-8, we have $\phi(0) \leq 0$, and by Theorem 5-4 this establishes the inequality (5-1) of the Milin conjecture.

6. Equality.

In this section, we complete the proof of the Milin conjecture by showing that equality only holds for rotations of the Koebe function. We again follow [FP].

Theorem. *Let $f \in S$, and suppose f is not a rotation of the Koebe function. Then there is strict inequality in the Milin conjecture.*

Proof. Write

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots$$

$$\log\frac{f(z)}{z} = 2\gamma_1 z + 2\gamma_2 z^2 + 2\gamma_3 z^3 + \ldots$$

By Bieberbach's Theorem (section 2), $|a_2| < 2$. This is the only time we use the fact that f is not a rotation of the Koebe function.

By Theorem 4-2, we can find $\{f_m\} \subseteq S'$ with $\{f_m\} \to f$ in the topology Δ. By the Löwner Representation Theorem, for each f_m we can choose a parameterized family $g_m(z,t)$ of univalent functions such that, for each non-negative real number t, the function

$$z \mapsto e^{-t}g_m(z,t) = z + a_{2,m}(t)z^2 + a_{3,m}(t)z^3 + \ldots \in S,$$

with $g_m(z,0) = f_m(z)$, and

$$\dot{g}_m(z,t) = \frac{1 + \kappa_m(t)z}{1 - \kappa_m(t)z} z g_m'(z,t)$$

for some continuous function

$$\kappa_m : [0,\infty) \to \left\{z \in \mathbb{C}\ \middle|\ |z| = 1\right\}.$$

Write

$$\log\frac{e^{-t}g_m(z,t)}{z} = c_{1,m}(t)z + c_{2,m}(t)z^2 + \ldots, \qquad |z| < 1.$$

Then since

$$\log\left(1 + a_{2,m}(t)z + ...\right) = a_{2,m}(t)z + ...,$$

we have $c_{1,m}(t) = a_{2,m}(t)$, for all m and for all $t \in [0, \infty)$.

Fix a positive integer n. We wish to show that

$$\sum_{k=1}^{n}\left(k|2\gamma_k|^2 - \frac{4}{k}\right)(n + 1 - k) < 0. \tag{6-1}$$

Since $|a_2| < 2$, we can find a real number α such that

$$|\, c_{1,m}(0)\,| = |\, a_{2,m}(0)\,| < \alpha < 2,$$

for all sufficiently large m. By Lemma 5-6 (with $k = 1$), for each m we have

$$\begin{aligned} |\, c'_{1,m}(t)\,| &= |\, c_{1,m}(t) + 2\kappa_m(t)\,| \\ &\leq |\, c_{1,m}(t)\,| + 2 \\ &\leq 4 \end{aligned}$$

by Bieberbach's Theorem. Hence for all sufficiently large m,

$$|\, c_{1,m}(t)\,| \leq |\, c_{1,m}(0)\,| + 4t < \alpha + 4t.$$

For each m, let

$$\phi_m(t) = \sum_{k=1}^{n}\left(k\,|\, c_{k,m}(t)\,|^2 - \frac{4}{k}\right)\tau_k(t).$$

Since $\lim_{m\to\infty} c_{k,m}(0) = 2\gamma_k$, (6-1) will follow from showing that $\lim_{m\to\infty}\phi_m(0) < 0$. From Lemma 5-7 and Theorem 5-5, it follows that

$$\begin{aligned} \phi'_m(t) &\geq |\, b_{1,m}(t) + 2\,|^2\left(\frac{-\tau'_1(t)}{1}\right) \\ &= |\, c_{1,m}(t)\kappa_m(t)^{-1} + 2\,|^2\left(-\tau'_1(t)\right) \\ &\geq (2 - \alpha - 4t)^2\left(-\tau'_1(t)\right), \end{aligned}$$

for sufficiently large m, provided $0 \leq t < \frac{2-\alpha}{4}$. Then using Theorems 5-1 and 5-8, we have $\phi_m(x) \leq 0$, and hence

$$\begin{aligned} \phi_m(0) &= \phi_m\left(\frac{2-\alpha}{8}\right) - \left(\phi_m\left(\frac{2-\alpha}{8}\right) - \phi_m(0)\right) \\ &\leq -\left(\phi_m\left(\frac{2-\alpha}{8}\right) - \phi_m(0)\right) \\ &= -\int_0^{\frac{2-\alpha}{8}} \phi'_m(t)\,dt \\ &\leq -\int_0^{\frac{2-\alpha}{8}} (2 - \alpha - 4t)^2\left(-\tau'_1(t)\right)\,dt \\ &\leq -\left(\frac{2-\alpha}{2}\right)^2\int_0^{\frac{2-\alpha}{8}}\left(-\tau'_1(t)\right)\,dt. \end{aligned}$$

Since $\alpha < 2$, it follows from Theorem 5-5 that this last expression is negative. Furthermore, it is independent of m. Hence, $\lim_{m\to\infty} \phi_m(0) < 0$, proving (6-1). □

REFERENCES

[**AG**] R. Askey and G. Gasper, *Positive Jacobi polynomial sums II*, Amer. J. Math. **98** (1976), 709-737.

[**Ba**] I.E. Bazilevich, *On distortion theorems in the theory of univalent functions*, Mat. Sb. **28(70)** (1951), 283-292. (Russian)

[**Bi**] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. (1916), 940-955.

[**dB1**] L. de Branges, *A proof of the Bieberbach conjecture*, Preprint E-5-84, Steklov Math. Institute, LOMI, Leningrad, 1984, 1-21.

[**dB2**] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137-152.

[**dB3**] L. de Branges, *Underlying concepts in the proof of the Bieberbach conjecture*, 25-42, in *Proceedings of the International Congress of Mathematicians*, A.M. Gleason, ed., Berkeley, August 3 - 11, 1986.

[**BDDM**] A. Baernstein II, D. Drasin, P.L. Duren, and A. Marden, *"Preface"*, *vii-xiv*, in *The Bieberbach Conjecture: Proceedings of the Symposium on the Occasion of the Proof*, Math. Surveys and Monographs No. 21, Amer. Math. Soc., 1986.

[**CS**] Z. Charzyński and M. Schiffer, *A new proof of the Bieberbach conjecture for the fourth coefficient*, Arch. Rational Mech. Anal. **5** (1960), 187-193.

[**Di**] J. Dieudonné, *Sur les fonctions univalentes*, C. R. Acad. Sci. Paris **192** (1931), 1148-1150.

[**Du1**] P.L. Duren, *Coefficients of univalent functions*, Bull. Amer. Math. Soc. **83** (1977), 891-911.

[**Du2**] P.L. Duren, *Univalent functions*, Springer-Verlag, Heidelberg and New York, 1983.

[**Fi**] C.H. FitzGerald, *Quadratic inequalities and coefficient estimates for schlicht functions*, Arch. Rational Mech. Anal. **46** (1972), 356-368.

[**FP**] C.H. FitzGerald and Ch. Pommerenke, *The de Branges theorem on univalent functions*, Trans. Amer. Math. Soc. **290** (1985), 683-690.

[**Go**] G.M. Goluzin, *Geometric theory of functions of a complex variable*, Gosudarst. Izdat., Moscow, 1952; English transl., Amer. Math. Soc., Providence, R.I., 1969.

[**Gro**] T.H. Gronwall, *Some remarks on conformal representation*, Ann. of Math. **16** (1914-1915), 72-76.

[**Gru**] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Functionene*, Math. Z. **45** (1939), 29-61.

[**GS1**] P.R. Garabedian and M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Rational Mech. Anal. **4** (1955), 683-690.

[**GS2**] P.R. Garabedian and M. Schiffer, *The local maximum theorem for the coefficients of univalent functions*, Arch. Rational Mech. Anal. **26** (1967), 1-32.

[**Ha1**] W.K. Hayman, *The asymptotic behaviour of p-valent functions*, Proc. London Math. Soc. **5** (1955), 257-284.

[**Ha2**] W.K. Hayman, *Multivalent functions*, Cambridge University Press, London and New York, 1958.

[**Ho**] D. Horowitz, *A refinement for coefficient estimates of univalent functions*, Trans. Amer. Math. Soc. **185** (1973), 265-270.

[**Li**] J.E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23** (1925), 481-519.

[**Lö1**] K. Löwner (C. Loewner), *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises* $|z| < 1$*, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden*, Ber. Verh. Sächs. Ges. Wiss. Leipzig **69** (1917), 89-106.

[**Lö2**] K. Löwner (C. Loewner), *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I*, Math. Ann. **89** (1923), 103-121.

[**LM**] N.A. Lebedev and I.M. Milin, *An inequality*, Vestnik Leningrad. Univ. **20** (1965), No. 19, 157-158. (Russian)

[**Mi1**] I.M. Milin, *Estimation of coefficients of univalent functions*, Dokl. Akad. Nauk SSSR **160** (1965), 769-771; English transl. in Soviet Math. Dokl. **6** (1965), 196-198.

[**Mi2**] I.M. Milin, *On the coefficients of univalent functions*, Dokl. Akad. Nauk SSSR **176** (1967), 1015-1018; English transl. in Soviet Math. Dokl. **8** (1967), 1255-1258.

[**Mi3**] I.M. Milin, *Univalent functions and orthonormal systems*, Izdat. "Nauka", Moscow, 1971; English transl., Amer. Math. Soc., Providence, R.I., 1977.

[**Ne**] R. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, Översikt av Finska Vetenskaps-Soc. Förh. **63(A)** (1920-21), no. 6, 1-21.

[**Oz**] M. Ozawa, *On the Bieberbach conjecture for the sixth coefficient*, Kōdai Math. Sem. Rep. **21** (1969), 97-128.

[**Pe**] R.N. Pederson, *A proof of the Bieberbach conjecture for the sixth coefficient*, Arch. Rational Mech. Anal. **31** (1968), 331-351.

[**Po**] Ch. Pommerenke, *Univalent functions*, Vandenhoech & Ruprecht, Göttingen, 1975.

[**PS**] R.N. Pederson and M. Schiffer, *A proof of the Bieberbach conjecture for the fifth coefficient*, Arch. Rational Mech. Anal. **45** (1972), 161-193.

[**Rob**] M.S. Robertson, *A remark on the odd schlicht functions*, Bull. Amer. Math. Soc. **42** (1936), 366-370.

[**Rog**] W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Math. Z. **35** (1932), 93-121.

[**Sz**] O. Szász, *Über Funktionen, die den Einheitskreis schlicht abbilden*, Jber. Deutsch. Math.-Verein. **42** (1933), 73-75.