Probabilities of Streaks in Online Chess

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Introduction:

Online chess has become extremely popular, especially on the web site Chess.com [1]. Such games have significant potential for cheating, since chess-playing computer programs are now much better than humans at chess and can be easily consulted (either manually or automatically) during a match to achieve superior play. Chess.com actively monitors and attemps to catch cheaters [2], and has recently published a detailed report of such matters [3]. Nevertheless, allegations of cheating continue, and are very controversial [4].

Some of the allegations of cheating concern long streaks of games which were all (or almost all) won by a specific player. In particular, concerns about cheating have been raised [4, 5] with regards to a recent streak of 46 high-level games played by the top-level player Hikaru Nakamura (player name: Hikaru), of which he won 45 and tied one.

I was asked by Erik Allebest and Dan Rozovsky of Chess.com to perform an independent statistical analysis of such winning streaks in chess play on the Chess.com online web site. To facilitate this, I was supplied [6] with data showing all results on Chess.com of seven different top-level players, including Hikaru.

In this report, I conduct a statistical examination of evidence of unusual or unlikely or surprising streaks in Hikaru's Chess.com game record. To do this, I first examine the nature of chess ratings, expected scores, win and tie probabilities, and game correlations, to establish a (simple, direct) model for the probabilities of various online chess outcomes.

Note, however, that the existence of unlikely streaks is a different matter from the issue of cheating. Indeed, a player who cheats in a consistent, regular manner might obtain a chess record which is indistinguishable from a stronger (but honest) player. Conversely, a player might perform considerably better over a short period due to higher concentration or motivation or preparation, even without any cheating. So, this report should be viewed as merely investigating the existence of unlikely streaks, not of investigating the broader issue of cheating in online chess.

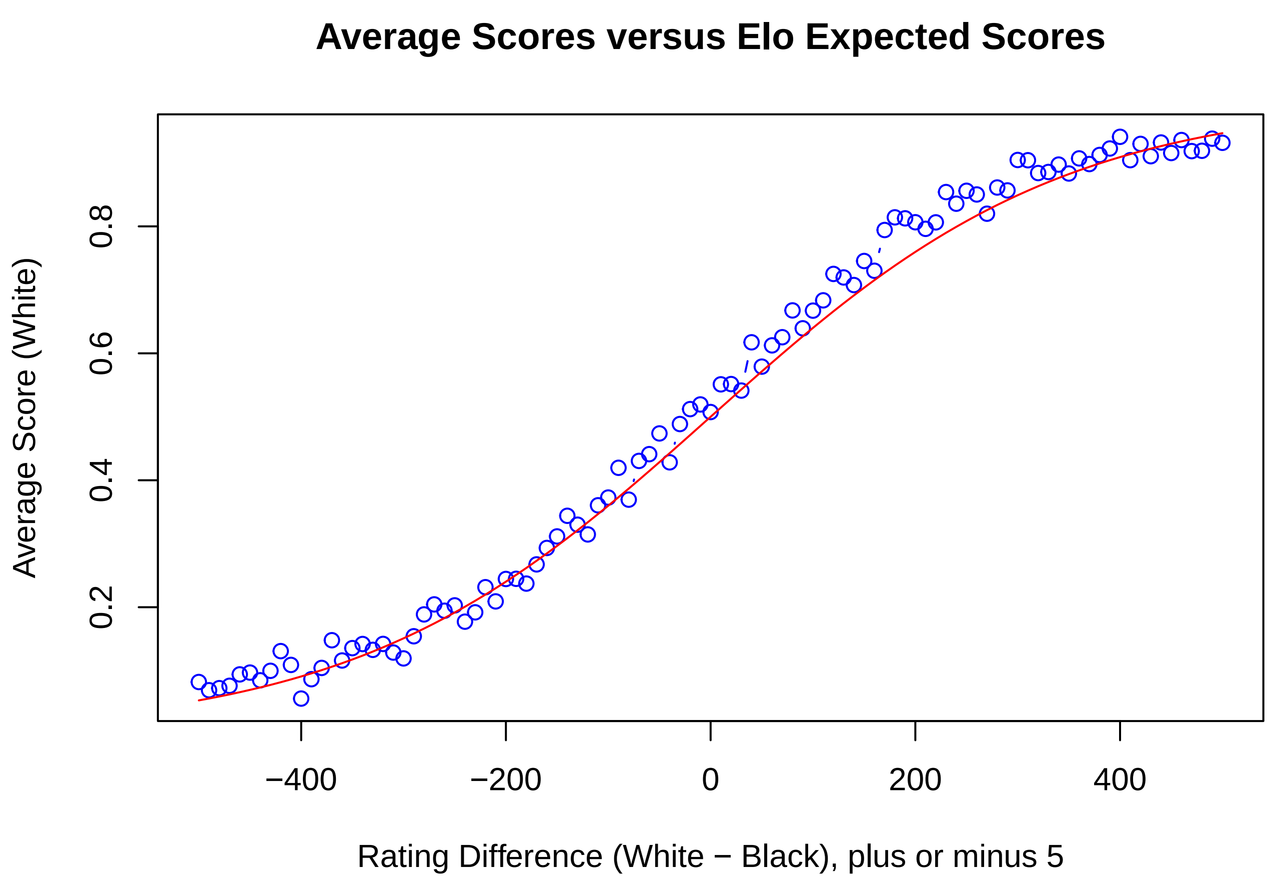
Chess Ratings and Expected Scores:

One way to assess probabilities of chess outcomes is through chess ratings. Every player on Chess.com is given an "Elo" chess rating, which is updated after each game. (Formally, these are “Chess.com blitz ratings”; many players also have blitz and classical ratings from the international chess federation FIDE, but we do not consider those here.)

In principle, these ratings should specify the expected score (i.e., average outcome) in each game, where the score is 1 for a win, 1/2 for a draw, or 0 for a loss. (The ratings do not, however, specify what fraction of the score should arise from wins versus draws; see next section.) Specifically, if White has rating A, and Black has rating B, so the rating difference is $D = A - B$, then White's expected score S should be given by the following simple Elo logistic formula:

$$\frac{1}{1+10^{-D/400}} = \frac{1}{1+10^{-(A-B)/400}}.$$

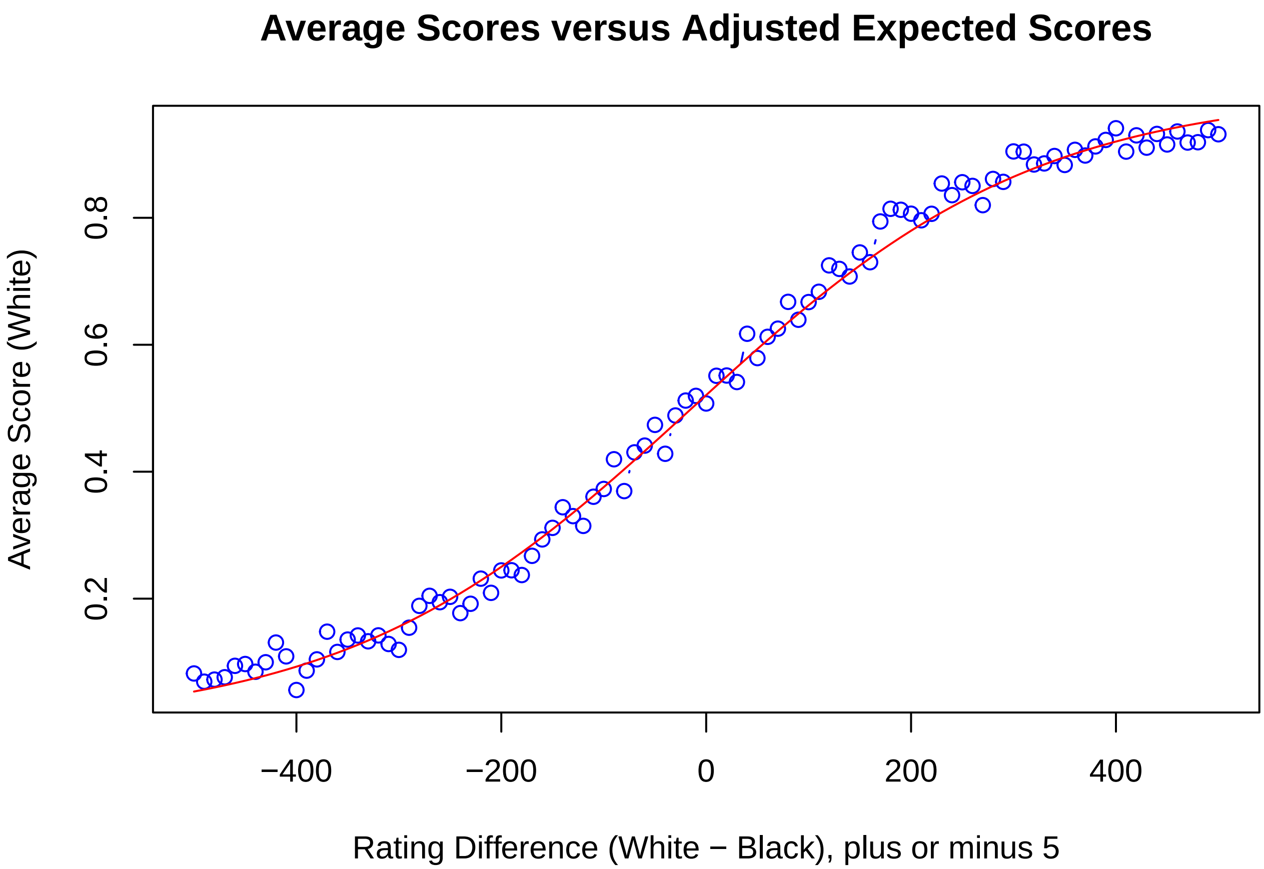
To test the validity of this formula, we combined all games in the seven data files, and “binned” them by rating difference into bin ranges $\ldots, (-14,-5), (-4,5), (6,15), (16,25), \ldots$. Then, for each bin, we computed the average score by White among all games within that bin. We then compared that to the curve specified above. The results were as follows:



This graph indicates that the average scores do approximately mimic the Elo expected scores, though with slight excess for rating differences between 100 and 300. Furthermore, they do not take into account the (small) advantage of playing White (i.e., moving first), even though the average score for White in the data is about 0.52 or slightly higher than 0.50. With a little bit of tweaking, we arrive at the slightly adjusted expected score

$$S = \frac{1}{1 + 10^{-(D+14)/390}} = \frac{1}{1 + 10^{-(A-B+14)/390}} \qquad (*)$$

(where the +14 represents White's advantage), which fits the data even better:



This fit appears to be accurate enough to use to estimate probabilities. So, in our analysis below, we assume the formula (∗) for White's expected score.

Draw (Tie) Probabilities:

In traditional chess tournaments with over-the-board games lasting many hours, draws (ties) are quite common. However, in online blitz chess they are less so; just 9.1% of the games in the dataset resulted in draws. Nevertheless, to evaluate the likelihood of long streaks of wins and draws, it is necessary to consider not just the expected score S, but more specifically the probability W of a win and probability T of a tie.

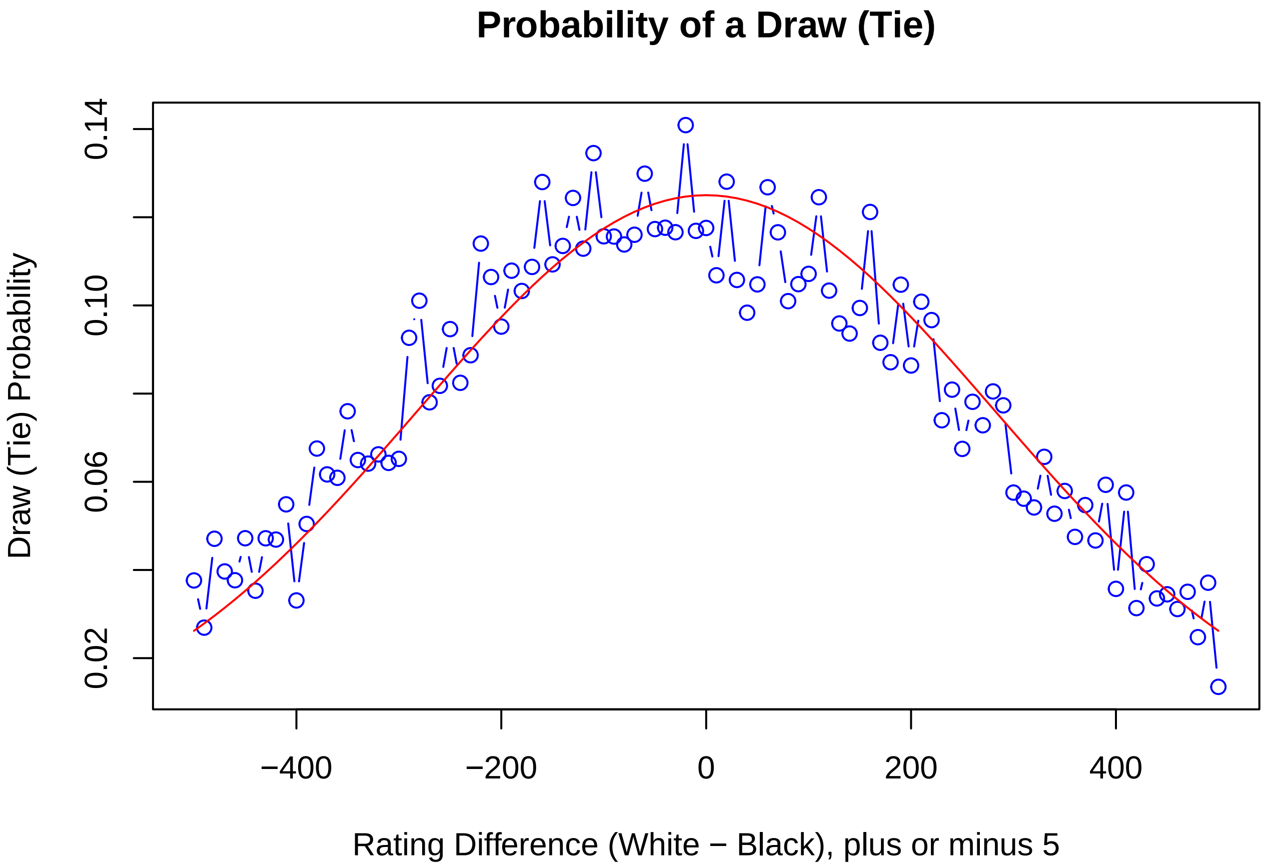
Since wins give a score of 1 while ties give a score of $\frac{1}{2}$, we must have

$$S = W + \frac{1}{2}T.$$

But how the total expected score S decomposes into the contribution W from wins and $\frac{1}{2}T$ from ties is unclear. Binning the data again as above, we observe that the simple exponential downward-quadratic function

$$T \;=\; (1/8)\, e^{-(D/400)^2} \tag{$**$}$$

gives a reasonably good approximation to the probability of a tie:



So, in our analysis below, we use the formula $(**)$ for the probability of a tie. It then follows from the above that the probability of a win is given by

$$W \;=\; S - \frac{1}{2}\,T$$

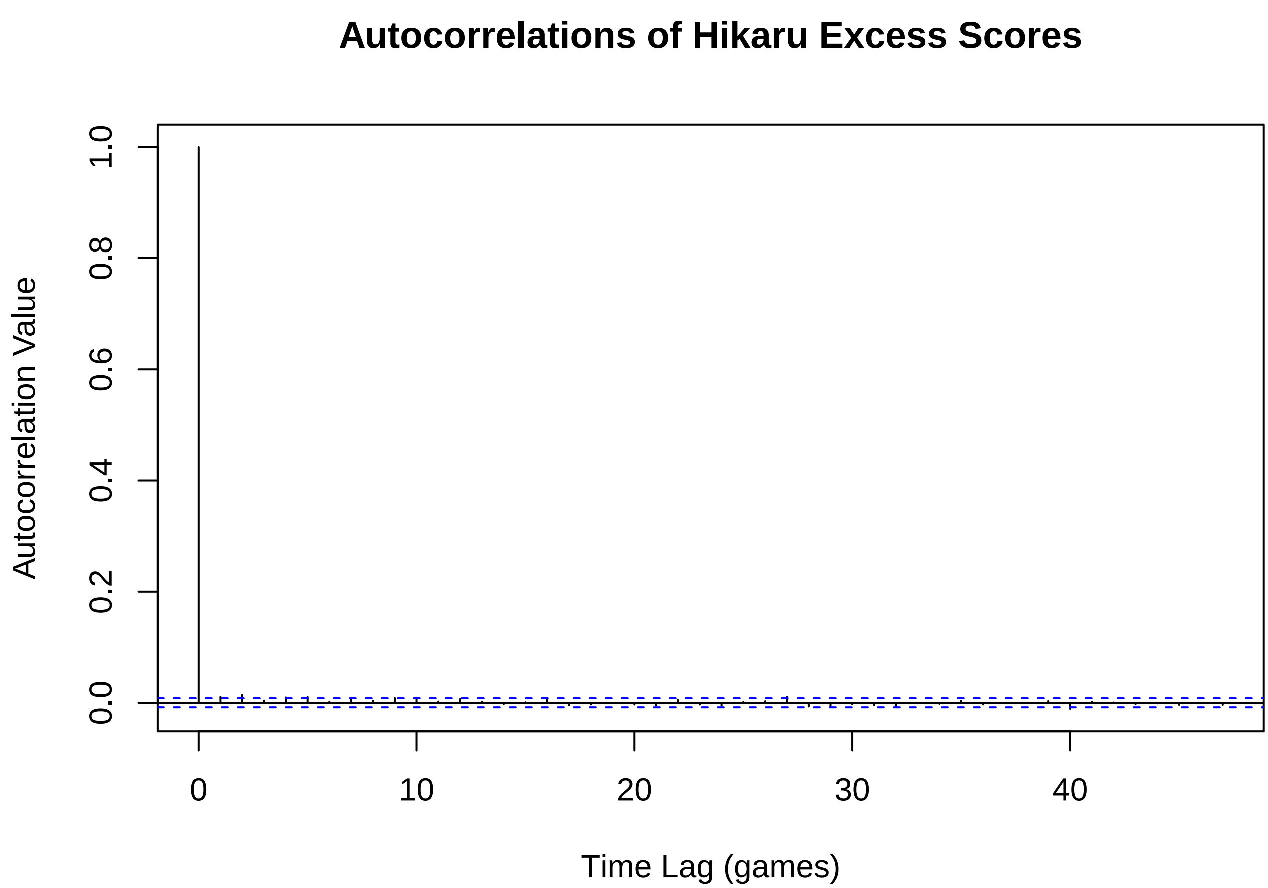
with S as in $(*)$ and T as in $(**)$.

Independence versus Hot Hand:

To model probabilities of streaks, another issue is the extent to which different games are independent. There is a long history of statistical debate about "hot hands" in basketball and other sports. It is quite plausible that there would be some "hot hand" or persistence of performance in chess games as well, especially for games played on the same day in rapid succession, perhaps even against the same opponent.

To investigate this, we examined the 57,421 games played by Hikaru on Chess.com (about which more below). For each game, we compute Hikaru's "excess score", defined as his actual score (i.e. 1 or 0 or $\frac{1}{2}$) minus his expected score E from $(*)$ (which depends on his rating difference with his opponent). This gives a time series list of excess score for all 57,421 games played by Hikaru on Chess.com, in order.

For such a time series, we can consider the "autocorrelations" which measure, for each time lag, the correlation between the excess score on games played at that spacing. For example, at lag=1, this measures the correlation of excess score between successive games. (The autocorrelation at lag=0 is always equal to one, since games have perfect correlation with themselves.) For Hikaru's list, the autocorrelations are as follows:



We see from this plot of autocorrelations that (to our surprise) the autocorrelations at all lags (except for lag=0) are all extremely close to zero. This indicates that there is virtually no correlation between Hikaru's excessive scores on successive games. That is, for these games at least, there is no overall evidence of a "hot hand", so they can be regarded as all being independent. We use this assumption in our analysis below.

(Of course, it is still possible there could still be some shorter-term hot hand effect, e.g. when playing a particularly intense match against a specific opponent, which we are not able to easily detect from the overall record. We do not consider that further here.)

Hikaru Winning Streaks – Identification:

Next, we investigate winning streaks in the Hikaru game data.

Hikaru is recorded as playing a total of 57,421 games on Chess.com over the date range 2014-01-06 to 2024-07-14, primarily against highly-rated opponents, primarily at time control 3m+0s (i.e., three minutes each for the entire game; 35,449 games) or 1m+0s (15,569 games) or 3m+1s (3,310 games). We combine all of these games together, in time order, to determine streaks. (It is also possible to separate out the games played at specific time controls, and/or against highly-rated opponents only; we have investigated this too, and the results are similar or less streaky compared to the below.)

To define a "streak", we have to decide how to handle draws. At one extreme, only wins continue a streak, while any draw or loss ends it. (We did investigate such "pure" winning streaks, and again the results are similar or less streaky compared to the below.) At the other extreme, wins or draws both continue a streak, while only a loss ends it. (This is a very loose definition, allowing many draws in a row to constitute a major "streak", so we do not consider it further.) As a compromise, since the most controversial of Hikaru's streaks involved just one draw, we use the "in-between" definition that a streak consists of a sequence of games with no losses and at most one draw. That is, a single draw continues a streak, but a second draw (or any loss) ends it.

With this definition, Hikaru has a total of 8,069 streaks (including some overlapping ones). Now, most of these are very short "streaks"; indeed 1,302 of them consist of just a single game. However, quite a few of them are reasonably large. Indeed, 226 of them are at least 30 games, and the largest are of lengths 121, 114, 107, 103, and 101.

Hikaru Winning Streaks – Probabilities:

Just because a streak is long, does not necessarily mean that it is unlikely. We need to measure the probabilities of each of Hikaru's streaks, to see which ones are most unlikely. We define the "raw" probability of each streak as the probability of his either winning all of those games (if he did win them all), or winning all or winning all but one and tying the other (if he did win all but one and tie one). This probability depends on the rating differences, according to the formulae $(*)$ and $(**)$. We assume independence between games; any persistence or "hot hand" assumption would instead make the streaks more likely. We can then compute the "raw" probability of that specific streak on those specific games.

Some very long streaks are not particularly unlikely. For example, Hikaru's streak of length 121 began with his game number 20,940, and took place on 22 December 2018 (except for the final game), with opponents having a mean rating of just 1,579 (compared to his rating of over 3,000 during that same period). His probability of scoring at least 120.5 on those 121 games then works out to 1/8.9, which is not particularly unlikely at all.

We thus focus specifically on Hikaru's longer streaks which are unlikely according to our probabilities. The following table shows all of his streaks which are "notable", i.e. at least 30 games long with raw probability less than one chance in 500:

Hikaru Notable Streaks (length 30+, and prob 1/500+):

line	streaknum	enddate	startgame	length	score	expected	1/prob
1	589	2016.04.06	7027	54	54	48.6	1766
2	2155	2018.07.01	18184	44	43.5	38	875.9
3	2414	2018.11.12	19665	40	39.5	33.9	1083.7
4	2415	2018.11.12	19666	57	56.5	49.2	9452.1
5	3734	2020.01.25	28227	30	29.5	23.9	1345.1
6	3917	2020.03.12	29340	41	40.5	33.3	11570.6
7	4465	2020.05.31	32790	61	61	55	4849.8
8	4551	2020.06.30	33483	53	52.5	47.1	808.1
9	7388	2023.11.17	51857	46	45.5	40	829.6
10	7770	2024.03.12	55162	35	34.5	29.5	568.4

Hikaru Notable Streaks – Analysis:

As can be seen from the above table, the most controversial streak, of length 46 ending on date 2023.11.17 (line 9) is not too far out of the ordinary. It has probability about one chance in 830. Indeed, a sequence of 57,421 games has about $57,421/46 \doteq 1,248$ different non-overlapping independent chances to achieve a streak of that length, so finding one with probability 1/830 is actually very likely.

Of the other streaks, just two are considerably less likely, namely lines 4 and 6 with probabilities 1/9,452 and 1/11,571 respectively. So, we consider them next.

The streak on line 4, with probability 1/9,452, consisted of 57 games during the period Nov. 10–12, 2018, at time controls 3m+0s (37 games) or 1m+0s (20 games). (As an aside, the previous streak on line 3 largely overlaps with this one, beginning with a draw one game earlier, and thus ending earlier upon the next draw.) This streak was played against a total of eight different opponents (none more than ten games each). The mean opponent rating was 2,786, which on average was 316 less than Hikaru's rating which hovered around 3,100.

Now, out of 57,421 games total, there are still over 1,000 different independent opportunities to establish a streak of length 57 (and much more if overlapping, non-independent intervals are considered). So, as a first approximation, suppose there are 1,000 independent opportunities to establish a streak, each of independent probability 1/9,452. Then the probability of achieving such a streak would be given by

$$1 - [1 - \frac{1}{9,452}]^{1,000} \doteq 0.100 = 1/10\,.$$

That is, under this independence approximation, the probability of achieving such a streak over the course of 57,421 games, is approximately one chance in ten. That is not particularly surprising, and does not even reach the usual 0.05 level for statistical significance. (And, this is just a *lower bound*, ignoring overlapping opportunities; see the next section.)

The streak on line 6, with probability 1/11,571, consisted of 41 games during the period March 10–12, 2020, all at time control 3m+0s. This streak was played against five different opponents, including Bigfish1995 (13 games) and ToivoK3 and wonderfultime (10 games each). The mean opponent rating was 3,008, which on average was about 250 less than Hikaru's rating which had climbed to around 3,250 by this point.

Out of 57,421 games total, there are about 1,400 different independent opportunities to establish a streak of length 41 (again ignoring overlapping opportunities). So, we consider the approximation that there are 1,400 independent opportunities to establish a streak, each of independent probability 1/11,571. Then the probability of achieving such a streak would be given by

$$1 - [1 - \frac{1}{11,571}]^{1,400} \doteq 0.114 \doteq 1/8.8\,,$$

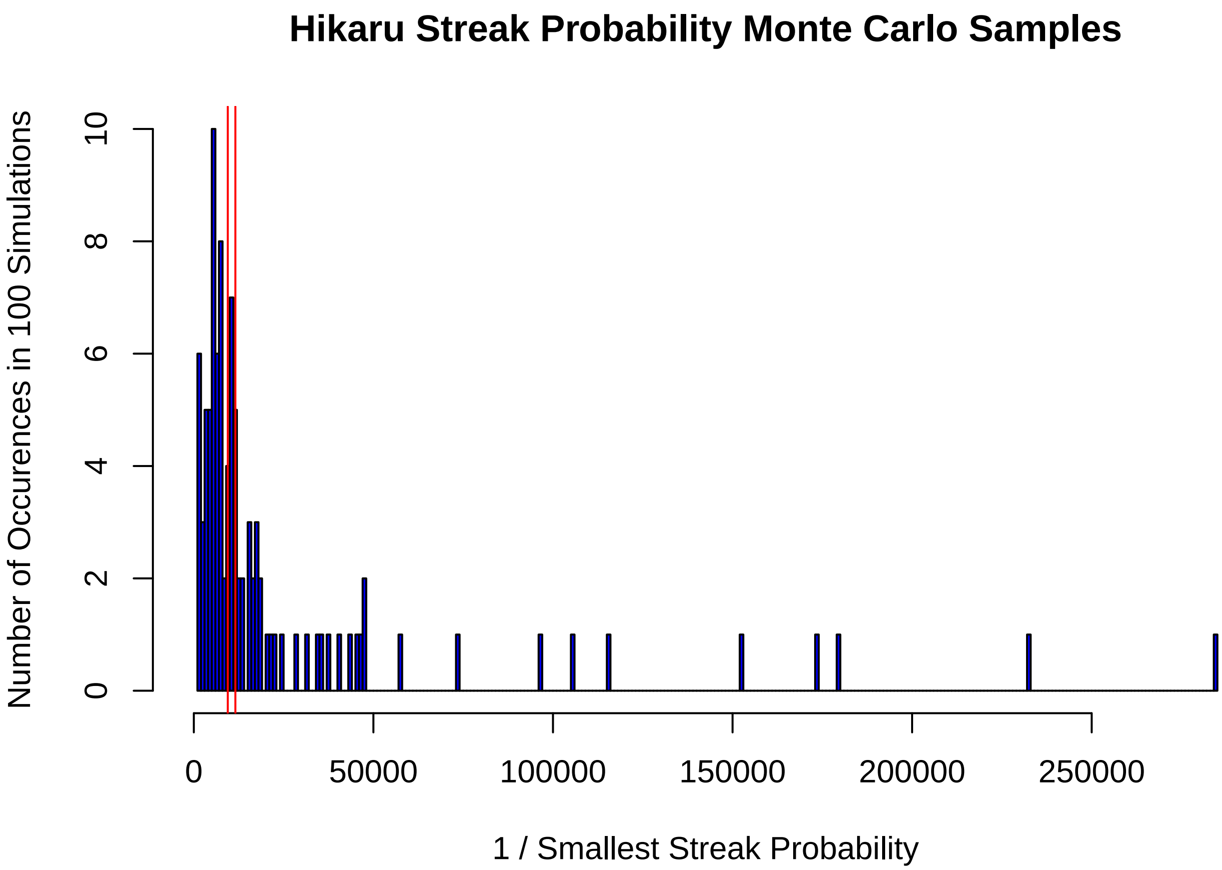
That is, under this (lower bound) approximation, the probability of achieving such a streak over the course of 57,421 games is about one chance in 8.8. That is not very surprising, and again does not reach the usual 0.05 level for statistical significance.

Hikaru Monte Carlo Simulation:

The above discussion indicates that Hikaru's individual win streaks are not particularly surprising. However, the approximate probabilities computed above are just lower bounds, since they do not take into account the additional possibilities of long streaks in overlapping (and hence not independent) game sequences.

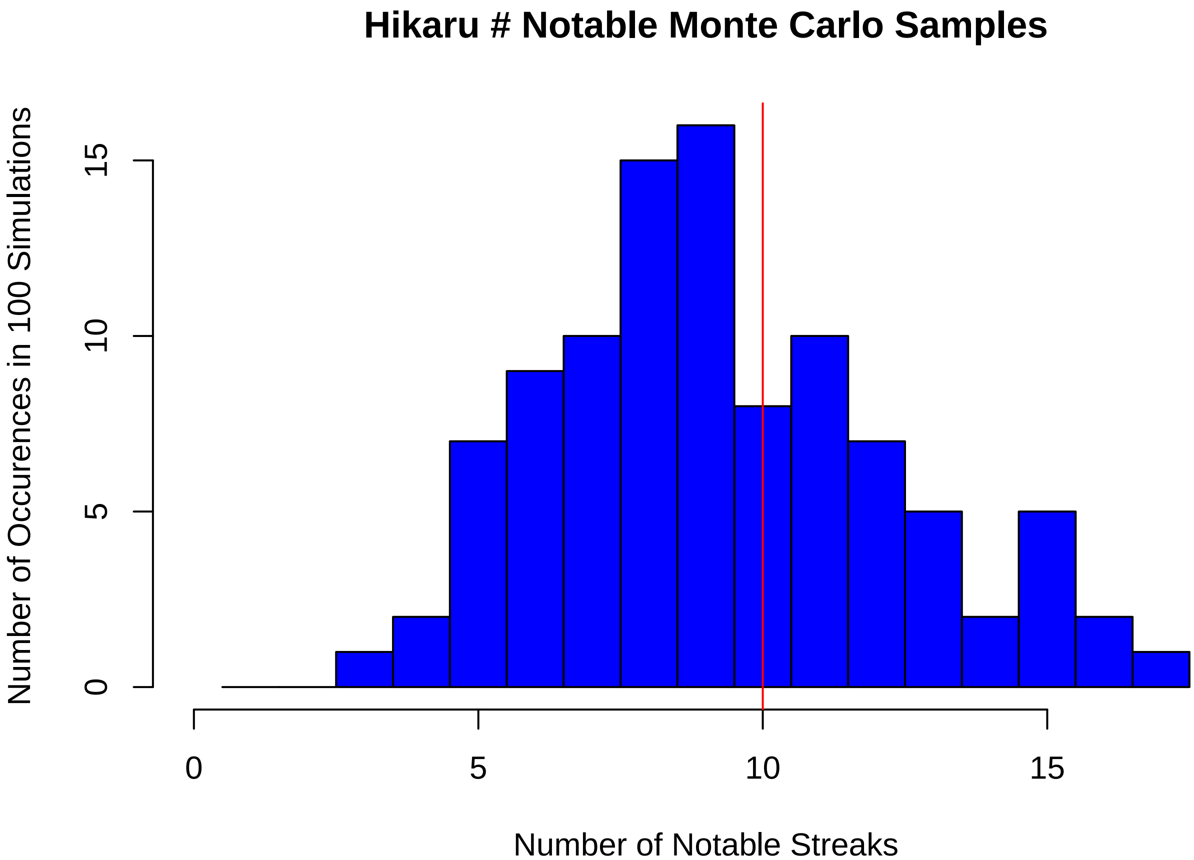
To analyse this further, we now conduct a Monte Carlo (random) simulation. Specifically, using the actual player ratings for each of Hikaru's 57,421 games, we simulated fresh independent results using the probabilities of wins and ties from ($*$) and ($**$). We repeated this simulation 100 different times, each time recording the smallest individual streak probability (i.e. the largest 1/probability), and also the total number of "notable" streaks as above (i.e. length at least 30, and 1/probability at least 500).

The distribution of the largest 1/probability, i.e. 1 / smallest-probability, in those 100 simulations, together with Hikaru's actual two largest 1/probability values (11,570.6 and 9,452.1, respectively) in red, is as follows:



We see from the graph that, while the actual values 11,570.6 and 9,452.1 are larger than many of the simulated maximum 1/probability values, there are also many simulated 1/probability values which are much larger than that. Indeed, the largest simulated 1/probability value is over 284,000, and the mean simulated 1/probability value is over 26,000, and even the median simulated 1/probability value is 10,461.92 which is close to Hikaru's 11,570.6 value. In fact, in 43 of the 100 simulations (nearly half), the least likely streak is less likely than the observed 1/11570.6 one. This further confirms that Hikaru's least likely streaks are not surprising over such a long collection of games.

Similarly, the distribution of the number of "notable" streaks (i.e. length at least 30, and 1/probability at least 500) in each of the 100 simulations, together with Hikaru's actual value of 10 in red, is as follows:

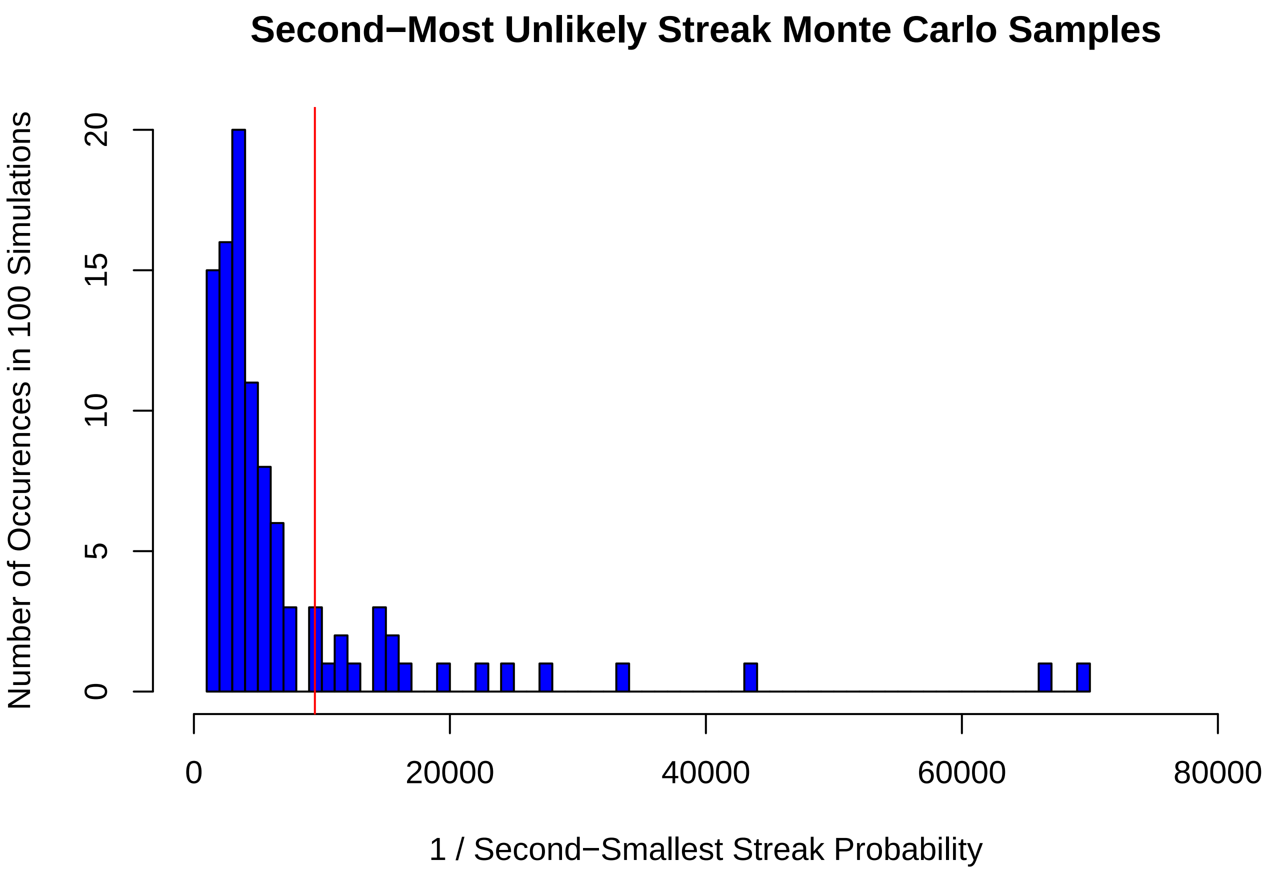


This graph shows that Hikaru's actual number 10 of notable streaks is quite typical for the simulations, which have a mean of 9.25 and median of 9. Indeed, in 40 of the 100 simulations, the number of notable streaks was equal to or greater than the observed 10. This confirms that Hikaru's number of notable streaks is also not surprising.

Second-Least-Likely Streaks:

As a final test, we note that Hikaru had two streaks whose raw probabilities were quite low, namely 1/11570.6 and 1/9452.1. As discussed above, each of these streaks on its own turns out to be not at all surprising. However, this raises the question of how unlikely it would be to have two such streaks which are *each* so unlikely.

To test this, we ran another Monte Carlo simulation, again simulating the possible outcomes of Hikaru's 57,421 games, but this time looking at the raw probability of the *second* most unlikely streak. Once again, we repeated this simulation 100 times. The resulting distribution, together with Hikaru's observed value of 1/9452.1, appears as follows:



This graph indicates that Hikaru's second-most-unlikely streak probability of 1/9452.1 is slightly less likely than typical. Indeed, only 18 of the 100 simulations had second-most-unlikely streaks which were less likely. Nevertheless, even 18% is still quite a large fraction, much larger than the 0.05 required for statistical significance. Indeed, some of the simulated second-most-unlikely streaks were considerably less likely, with two of their probabilities less than 1/65,000. This analysis indicates that Hikaru's second-most-unlikely streak was slightly less likely than expected, but still well within the usual range of statistical fluctuation.

Summary:

This statistical analysis indicates that Hikaru's online chess winning streaks are not particularly surprising. His recent controversial streak of length 46 is well within expected levels. His two most surprising streaks are of length 57 in 2018, and of length 41 in 2020. Although their raw probabilities are each about one chance in 10,000, the probability of observing each such streak over the course of so many games is still shown to be above 10% based on independent non-overlapping opportunities alone, and about 43% in Monte Carlo simulations, and hence not unlikely. Having *two* such notable streaks is somewhat less likely, but still occurs about 18% of the time, well within usual statistical variability. Overall, the streaks observed in Hikaru's Chess.com record are fairly typical given the ratings of the players over Hikaru's long record of games.

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