SCI 199Y: Random Walks and Mathematical Discovery

Math reminder, week 10.

Recall that we were considering the following game (to be referred to as the original game). **A** starts with a pennies, and **B** starts with $8-a$ pennies. A fair 6-sided die is repeatedly rolled. If it comes up 1 or 2, then **B** gives one penny to **A**. If it comes up 3, 4, 5, or 6, then **A** gives one penny to **B**. This is repeated until either **A** or **B** wins all the pennies. That person is the "winner". Recall that we wrote $s(a)$ for the chance that **A** wins this game, starting with a pennies.

What follows is a "reminder" of how we derived a formula for $s(a)$.

Step #1. Obviously $s(0) = 0$ and $s(8) = 1$.

Step #2. By considering what happens on the *first* bet, we see that

$$s(a) = \frac{1}{3} s(a+1) + \frac{2}{3} s(a-1)\,,$$

for $a = 1, 2, 3, 4, 5, 6, 7$.

Step #3. Since $s(a) = \frac{1}{3}s(a) + \frac{2}{3}s(a)$, this last formula can be re-written as

$$s(a+1) - s(a) = 2\left(s(a) - s(a-1)\right)\,.$$

Step #4. Hence, setting $x = s(1)$, we see that

$$s(1) - s(0) = x, \quad s(2) - s(1) = 2x, \quad s(3) - s(2) = 4x, \quad \text{etc.}$$

and in general that

$$s(a+1) - s(a) = 2^a x$$

for $a = 0, 1, 2, 3, 4, 5, 6, 7$.

Step #5. It follows that, for $a = 0, 1, 2, 3, 4, 5, 6, 7, 8$,

$$s(a) = s(a) - s(0) = (s(a) - s(a-1)) + (s(a-1) - s(a-2)) + \ldots + (s(1) - s(0))$$

$$= \left(2^{a-1} + 2^{a-2} + \ldots + 2 + 1\right) x = \left(2^a - 1\right) x\,.$$

Step #6. Since $s(8) = 1$, it follows that $x = \frac{1}{2^8 - 1}$, whence

$$s(a) = \frac{2^a - 1}{2^8 - 1} = \frac{2^a - 1}{255}\,.$$