SCI 199Y: Random Walks and Mathematical Discovery

Math exercise, weeks 15 and 16 (at least).

On this assignment we shall study mathematically two properties of "simple symmetric random walk".

You will be divided into "math-homogeneous groups", as follows:

- **A:** Jeff, Joe, Edgar, Andrew, any other very mathematically inclined & interested students;
- **B:** Gabrielle, Glenn, Michelle, any other less mathematically inclined/interested students;
- **C:** Alison, Edward, Braveen;
- **D:** Jessie, Kyla, Bobby, Kathryn;
- **E:** Denny, Alex, Roshan, Carol-Ann.

[Note: These groups were put together based on your responses on the index card last week. They are tentative groupings; if you would like to change groups then please speak to me.]

With your group, work on the following questions. [My intention is that most groups will concentrate on the "Mathematical Questions" below, but that the less mathematically inclined group might concentrate on the "Philosophical Questions", and the more mathematically inclined group might possibly get as far as the "Advanced Questions". As a group please work at your own pace on the questions which you find appropriate.] Note that this assignment contains multiple pages, and may take several weeks.

Recall that the rules of simple symmetric random walk are as follows. We place a penny at the box labeled "0". We then continually roll a fair 6-sided die. Each time the die comes up 4, 5, or 6, we move the penny one box to the right. Each time the die comes up 1, 2, or 3, we move the penny one box to the left.

Last week we made two "conjectures". To state them, let's let "S_n" stand for the position of the penny after n moves. (Thus, S_0 is always 0, S_1 is always either 1 or -1, S_4 is the position after 4 rolls whatever that is, S_{10} is the position after 10 rolls, etc.).

Conjecture #1. The average value of S_n will be 0.

Conjecture #2. The average value of $(S_n)^2$ will be n.

Mathematical Questions.

Let us introduce some "random variables" X_1, X_2, X_3, X_4, ..., defined as follows. If the i^{th} roll is a 4, 5, or 6, then $X_i = 1$. If the i^{th} roll is a 1, 2, or 3, then $X_i = -1$. For example, if the first four rolls are 5, 6, 2, 4, then we would have $X_1 = 1$, $X_2 = 1$, $X_3 = -1$, $X_4 = 1$.

(1) If the following three rolls were 4, 2, 1, then what would X_5, X_6, and X_7 be?

(2) In terms of the quantities $X_1, X_2, X_3, \ldots$, what does S_1 equal? What does S_2 equal? What about S_4? S_{10}? S_{1000}? S_n? [Just in case you're stuck: The first two answers are $S_1 = X_1$, and $S_2 = X_1 + X_2$. Why?]

(3) In terms of the quantities $X_1, X_2, X_3, \ldots$, what does $(S_1)^2$ equal? What does $(S_2)^2$ equal? What about $(S_4)^2$? $(S_n)^2$?

Now let's see how these new random variables can help us to compute average values.

(4) What do you think the average value of X_1 is? Why do you think that? Try to justify your answer as well as you can.

(5) What is the average value of X_2? of X_{10}? of X_i for any choice of i?

(6) Putting together the results of questions (2) and (5), try to explain why Conjecture #1 is true.

Now to establish Conjecture #2:

(7) What is the average value of $(X_1)^2$? of $(X_2)^2$? of $(X_i)^2$?

(8) What is the average value of X_1X_2? of X_1X_3? of X_4X_9? of X_iX_j when $i \neq j$?

(9) Putting together the results of questions (3), (7), and (8), try to explain why Conjecture #2 is true.

Philosophical Questions.

(10) Last week we didn't use any mathematical notation at all. But on this assignment, we're using symbols like S_n and X_i. Why was it necessary to introduce mathematical notation on this assignment?

(11) What are some advantages and disadvantages of using mathematical notation?

(12) Why are quantities like X_1, X_2, etc. called "random variables"? Do you think that, say, S_4 is also a random variable?

(13) What does it mean to call something a "Conjecture"? Once we have a conjecture, why do we need to do more work?

(14) What would it take to *convince* you that the two Conjectures are really true? Would a mathematical proof help?

(15) How could the mathematics be presented in a way that would make it <u>easier</u> for you to think about?

(16) What does the "n" stand for in "S_n"? What does the "i" stand for in "X_i"? Why didn't we use the same letter for both of them?

(17) When doing mathematical work, is it useful to consider "philosophical questions" such as these? Do you think that mathematical geniuses consider such things?

(18) Have you <u>enjoyed</u> considering these philosophical questions? What suggestions do you have for how I can make your mathematical work more enjoyable in the future?

<u>Advanced Questions.</u>

(19) In the expression for $(S_n)^2$ developed in question (3), was the expression more useful in simplified form, or when multiplied out? Will this *always* be the case, in *any* mathematical exercise? Discuss.

(20) In questions (4), (5), (7), and (8), we use the expression "average value". [Note: The more correct technical term is *expected value*.] What does that expression really *mean*? Give as precise and complete an answer as possible.

(21) To complete question (6) – and again to complete question (9) – we had to use some general "property" of expected values. What is this property? How can we be sure that the property holds? Be as precise as possible.

(22) To complete question (8), were there any properties of expected value that were at work? Again, be as precise as possible.

Note: We have not yet discussed the probability that the random walk will return exactly to 0. We will discuss that in a few weeks. So, be sure to remember your "data" for this (i.e., what fraction of the time the random walk from last week had returned to 0 after 4 rolls, and after 10 rolls) until then.