SCI 199Y: Random Walks and Mathematical Discovery

Math exercise, weeks 19 and 20 (at least).

On this assignment we shall study return probabilities of simple symmetric random walk.

Recall that the rules of simple symmetric random walk are as follows. We place a penny at the box labeled "0". We then continually roll a fair 6-sided die. Each time the die comes up 4, 5, or 6, we move the penny one box to the right. Each time the die comes up 1, 2, or 3, we move the penny one box to the left.

Recall that on the last mathematical assignment, we decided that, if the position of the walk after n rolls is written as S_n, then

$$S_n = X_1 + X_2 + \ldots + X_n ,$$

where each X_i is either $+1$ (if the i^{th} roll is 4,5,6) or -1 (if the i^{th} roll is 1,2,3). On this assignment, we shall study the probability that $S_n = 0$; this probability can be written as $\mathbf{P}(S_n = 0)$.

With your mathematics group, work on the following questions. [My intention is that most groups will concentrate on the "Mathematical Questions" below, but that the less mathematically inclined group might concentrate on the "Experimental Question" and "Philosophical Questions", and the more mathematically inclined group might possibly get as far as the "Advanced Questions". Also, I hope that all groups will at least *look at* the "Follow-up Questions".] Note that this assignment contains multiple pages, and may take several weeks.

Preliminary, whole-class exercise.

(1) We shall begin by tabulating (on the blackboard) the results of your experiments from Week 14, about the probability of returning to 0 after 4 rolls, and after 10 rolls. What do you think the "true" probabilities are? What do you think will happen for different values of n?

Mathematical Questions.

(2) What is the probability that $S_1 = 0$? That $S_3 = 0$?

(3) What is the probability that $S_n = 0$, if n is *odd*?

(4) What is the probability that $S_2 = 0$? [Hint: If $S_2 = 0$, then it must be that *either* $X_1 = +1$ and $X_2 = -1$, *or* that $X_1 = -1$ and $X_2 = +1$. What are the probabilities of each of these two events?]

(5) What is the probability that $S_4 = 0$? that $S_6 = 0$? [Hint: These questions are more difficult than the previous one, but they can be answered by similar reasoning.]

(6) What is the probability that $S_n = 0$, when n is *even*? [Hint: This question is a bit trickier. It can be studied like the previous questions, but requires more general thinking. You will need to use the "binomial coefficient" or "choose formula" ...]

Experimental Question.

(7) Using a penny and a die, plus a 21-box sheet like for the experiment of Week 14, experiment with random walk to try to *estimate* the probabilities in questions (2), (4), and (5) above. [Hint: run the random walk over and over again, like in Week 14 ... and keep track of the fraction of times that it returns to 0 after the 2^{nd}, 4^{th}, and 6^{th} roll. How many trials do you think you should do to get an accurate estimate of the probability?]

Advanced Questions.

(8) What "happens" to the probability that $S_n = 0$, when n is large (and even)? Does it converge to 0? How quickly? In other words, try to find an *approximation* to this probability which makes more clear its behaviour when n is large. [Hint: You may wish to use *Sterling's Formula*, which says that for a large integer m, we have

$$m! \approx \left(\frac{m}{e}\right)^m \sqrt{2\pi m},$$

where $e = 2.71828\ldots$]

(9) Suppose we run two different simple symmetric random walks at the same time (each on a different playing board). What is the probability that they will *both* return to 0 after n rolls?

(10) More generally, suppose we run d different simple symmetric random walks at the same time (each on a different playing board). What is the probability that they will *all* return to 0 after n rolls?

(11) What "happens" to the probability in the previous question, when n is large and even? In other words, find an approximation to this probability, similar to question (8) above.

Follow-up Questions.

You may wonder why we care about these probabilities of returning to 0. One reason is the following. It is a fact that for any random process, the probability that it will eventually return to 0 is given by

$$\frac{\sum_{n=1}^{\infty} \mathbf{P}(S_n = 0)}{1 + \sum_{n=1}^{\infty} \mathbf{P}(S_n = 0)} .$$

Thus, the probability of eventually returning to 0 will be one **if and only if** the sum $\sum_{n=1}^{\infty} \mathbf{P}(S_n = 0)$ is **infinite**.

(12) For simple symmetric random walk, is this sum infinite? In other words, is there probability 1 that this random walk will eventually return to 0? [Hint: the approximation in question (8) may help.]

(13) If we instead run d different simple symmetric random walks at the same time, will the corresponding sum be infinite? In other words, will there be probability 1 that these d different walks will eventually *all* return to 0 at the same time? [Hint: the approximation in question (11) may help.] Does your answer depend on d? Do you find the result surprising?

Philosophical Questions.

(14) On this assignment, we're using mathematical symbols like S_n and X_i. Do they mean the same thing that they did on the last assignment? Is that an important issue?

(15) What is the relationship between the Experimental Question and the corresponding Mathematical Questions? Is one more important than the other? Is it necessary to try to answer both of them?

(16) Why do you think I included the "Follow-up Questions" on this assignment?

(17) Can you understand what the Follow-up Questions are they asking? What is your guess of what the answers will be?

(18) In the preamble to the Follow-up Questions, I gave an equation for the probability of a random process *eventually* returning to 0. However, I didn't prove this equation, or even explain why it might be true – I just said "it is a fact". Why do you think I did this? Do you think it was a good idea?