SCI 199Y: Random Walks and Mathematical Discovery

Math exercise, week 23 (at least).

On this assignment we shall continue our study of return probabilities for simple symmetric random walk. With your mathematics group, work on the following questions.

Recall that the rules of simple symmetric random walk are as follows. We place a penny at the box labeled "0". We then continually roll a fair 6-sided die. Each time the die comes up 4, 5, or 6, we move the penny one box to the right. Each time the die comes up 1, 2, or 3, we move the penny one box to the left.

Recall that the position of the walk after n rolls is

$$S_n = X_1 + X_2 + \ldots + X_n \,,$$

where each X_i is either $+1$ (if the i^{th} roll is 4,5,6) or -1 (if the i^{th} roll is 1,2,3).

We have seen that $\mathbf{P}(S_n = 0) = 0$ if n is odd. For n even, most groups have figured out that

$$\mathbf{P}(S_n = 0) \;=\; \frac{\binom{n}{n/2}}{2^n} \;=\; \frac{n!}{(n/2)!\,(n/2)!\,2^n}\,.$$

The reason is: If we list all possible sequences of n different $+1$'s and -1's, then there are 2^n such sequences, and $\binom{n}{n/2}$ of them have exactly $n/2$ different "$+1$" (and hence also $n/2$ different "-1").

[For example, if $n = 1000$, then $\mathbf{P}(S_{1000} = 0) = \frac{1000!}{500!\,500!\,2^{1000}} \doteq 0.0252150.$]

Mathematical Questions.

(1) Make sure everyone in your group understands why $\mathbf{P}(S_n = 0) = \frac{\binom{n}{n/2}}{2^n}$, when n is even.

(2) It is hard to figure out many "properties" of the above formula, because the factorials are difficult to work with. (For example: Does it converge to 0 as $n \to \infty$? How quickly?) Derive a simpler approximation for this formula. Use Sterling's approximation to the factorial, which says that for large even n,

$$n! \;\approx\; \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \qquad \text{and} \quad (n/2)! \;\approx\; \left(\frac{(n/2)}{e}\right)^{(n/2)} \sqrt{2\pi(n/2)}$$

(where $e = 2.71828\ldots$).

[Note: The resulting approximation is pretty good! For example, it says that $\mathbf{P}(S_{1000} = 0) \approx 0.0252313$.]

(3) It is a fact that for any random process, the probability of eventually returning to 0 is 100% **if and only if** the sum $\sum_{n=1}^{\infty} \mathbf{P}(S_n = 0)$ is **infinite**. Can we apply this result to simple symmetric random walk? Will this sum be infinite for the probabilities discussed in question (1)? [Hint: Use your approximation from question (2)! Also: It is a fact that if $C > 0$ and $\alpha > 0$ are constants, then $\sum_{n=1}^{\infty} C/n^{\alpha}$ is infinite for $\alpha \leq 1$, and finite for $\alpha > 1$. Thus, $\sum_{n=1}^{\infty} C/\sqrt{n} = \infty$, $\sum_{n=1}^{\infty} C/n = \infty$, $\sum_{n=1}^{\infty} C/n\sqrt{n} < \infty$, $\sum_{n=1}^{\infty} C/n^2 < \infty$, etc.]

Advanced Questions.

(4) Suppose we run two different simple symmetric random walks at the same time (each on a different playing board). What is the probability that they will *both* return to 0 after n rolls? [Hint: You shouldn't have to do much more calculation, just think about the probability that two different things both happen.]

(5) More generally, suppose we run d different simple symmetric random walks at the same time (each on a different playing board). What is the probability that they will *all* return to 0 after n rolls?

(6) Answer question (2) above, for the case of d different simple symmetric random walks. [Hint: Again, you shouldn't have to do much more calculation.]

(7) Answer question (3) above, for the case of d different simple symmetric random walks. Does your answer depend on the value of d? Do you find it surprising?

Philosophical Questions.

(8) Can you understand what questions (3) and (7) are asking? What is your guess of what the answers will be?

(9) Twice on this assignment, I say "it is a fact that ...", and then say something without proving it or explaining why it's true. For which two facts do I say this? Why do you think I did this? Do you think it was a good idea?