SCI 199Y: Random Walks and Mathematical Discovery

Group math exercise, week 5.

This week we will continue the process of solving mathematically the probabilities that came up in the betting game. Return to your same groups from the math exercise from last time. Re-introduce yourselves.

Recall that we were considering the following game. **A** starts with a pennies, and **B** starts with $8 - a$ pennies. A fair 6-sided die is repeatedly rolled. If it comes up 1 or 2, then **B** gives one penny to **A**. If it comes up 3, 4, 5, or 6, then **A** gives one penny to **B**. This is repeated until either **A** or **B** wins all the pennies. That person is the "winner".

Recall from last time that we wrote $s(a)$ for the chance that **A** wins this game, starting with a pennies. We saw that $s(0) = 0$ and $s(8) = 1$.

Working cooperatively with your group members, consider the following questions. (If you get stuck as a group, then that might be a good time to assign some group roles – such as Problem Restater, Strategy Suggester, and Checker – as suggested by Johnson & Johnson, p. 17.)

Question #1: At the end last time, we derived the equation

$$s(6) \ = \ \frac{1}{3}s(7) + \frac{2}{3}s(5)\,,$$

or equivalently that $s(6) = s(5) + \frac{1}{3}\,(s(7) - s(5))$. (We used a hint which read, "Imagine starting with $a = 6$ pennies for **A**, and think about the possibilities for what happens on the first bet.") Work together as a group to make sure that all group members understand exactly why this equation is true (so that you would be able to explain it to someone else!).

Question #2: More generally, find an equation relating the three numbers $s(a-1)$, $s(a)$, and $s(a+1)$, for any integer a between 1 and 7. (Question #1 corresponds to the case $a = 6$.)

Question #3: Summarize everything we know about the "unknown" quantities $s(0), s(1), s(2), \ldots, s(8)$. Do you think we now have enough information to find a formula for them (even if we're not quite sure how to do that)?

If you have time, then you may continue:

Question #4: Can you convert your equation relating $s(a-1)$, $s(a)$, and $s(a+1)$ (from Question #2), to an equation relating "$s(a+1) - s(a)$" to "$s(a) - s(a-1)$"? (**Hint:** It might help to note that, obviously, $s(a) = \frac{1}{3}s(a) + \frac{2}{3}s(a)$.)

Question #5: (still hard!) Putting all of this together, can you find a formula for $s(a)$?