STA 3047F, Fall 2001: Homework #1

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Due in Sidney Smith Hall room 6024 by 4:00 p.m. on Monday October 22, 2001.

Total variation distance

1. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be finite state spaces. Let μ_1 and ν_1 be probability distributions on $\mathcal{X}_1$, and let μ_2 and ν_2 be probability distributions on $\mathcal{X}_2$. Then $\mu_1 \times \mu_2$ is a probability distribution on $\mathcal{X}_1 \times \mathcal{X}_2$, defined by $\mu_1 \times \mu_2\ (x, y) = \mu_1(x)\mu_2(y)$ (and similarly for $\nu_1 \times \nu_2$).

Prove that

$$\|\mu_1 \times \mu_2 \ - \ \nu_1 \times \nu_2\|_{\text{var}} \ \leq \ \|\mu_1 - \nu_1\|_{\text{var}} \ + \ \|\mu_2 - \nu_2\|_{\text{var}} \, .$$

2. Let P be a transition matrix on a state space $\mathcal{X}$, with initial distribution μ_0 and stationary distribution π. Prove that the total variation distance to stationarity is *weakly decreasing,* in the sense that for any $k \geq 0$,

$$\|\mu_{k+1} - \pi\|_{\text{var}} \ \leq \ \|\mu_k - \pi\|_{\text{var}} \, .$$

Eigenvalue bounds

3. For each of the following transition matrices, determine (with explanation!) whether $\lambda_* = 1$ or $\lambda_* < 1$.

(a)

$$P \ = \ \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0.01 & 0.01 & 0.98 \end{pmatrix}$$

(b)

$$P \ = \ \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$P \ = \ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Random walks on finite abelian groups

4. Let $\mathcal{X} = \{0, 1, 2, 3\}$, and let

$$P = \begin{pmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.4 & 0.2 & 0.4 & 0 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0.4 & 0 & 0.4 & 0.2 \end{pmatrix} .$$

Let $\mu_0(0) = 1$ give the initial distribution. Find an exponentially decreasing bound on the variation distance $\|\mu_k - \pi\|$. (Hint: This is a random walk on a group!)

5. *Random walk with two-point support.* Consider again random walk on $\mathbf{Z}/(n)$, but with n arbitrary. Let $j \in \mathbf{Z}/(n)$ with $j \neq 0$, and define the step distribution by $Q(0) = Q(j) = 1/2$.

(a) Under what conditions on n and j will we have $\lambda_* = 1$?

(b) For what values of n can we be sure that $\lambda_* < 1$, for *any* non-zero value of j?

6. *Numerical bounds on the circle.* Consider random walk on $\mathbf{Z}/(n)$, with $n = 100$. For each of the following step distributions $Q(\cdot)$, find (i) $P(99, x)$ for all $x \in \mathbf{Z}/(n)$, (ii) $|\lambda_m|$ for $0 \leq m \leq 99$ (simplified if possible), and (iii) λ_*. (Note: λ_m may be a complex number!) Finally, (iv) find a value of k so that $\|\mu_k - \pi\|_{\text{var}} \leq e^{-20}$. (Warning: Don't forget to compute appropriate cosines in radians, not in degrees!)

(a) $Q(0) = Q(1) = 1/2$, with $Q(x) = 0$ otherwise.

(b) $Q(3) = Q(4) = 1/2$, with $Q(x) = 0$ otherwise.

(c) $Q(-3) = Q(3) = 1/4$, and $Q(0) = 1/2$, with $Q(x) = 0$ otherwise.

7. *Explicit $O(n^2)$ bounds.* Consider random walk on the state space $\mathcal{X} = \mathbf{Z}/(n)$, the integers mod n, with initial distribution given by $\mu_0(0) = 1$, and with step distribution given by $Q(0) = 0.9$, $Q(1) = 0.1$.

Find, with proof, *explicit* constants $A, B, \alpha, \beta, k_0, n_0 > 0$ (independent of n and k) such that

$$A\, e^{-\alpha k/n^2} \;\leq\; \|\mu_k - \pi\| \;\leq\; B\, e^{-\beta k/n^2}\,, \qquad \text{for all } \; n \geq n_0\,,\; k \geq k_0\, n^2\,.$$

In particular, this proves that this random walk takes $O(n^2)$ steps to converge.

8. *Convergence in a constant number of steps.* Consider random walk on $\mathcal{X} = \mathbf{Z}/(n)$ with $\mu_0(0) = 1$. Let n be a multiple of 4, and set $Q(x) = 1/(1 + (n/2))$ for $x =$

$-n/4, -(n/4)+1, \ldots, -1, 0, 1, 2, \ldots, (n/4)-1, n/4$, with $Q(x) = 0$ otherwise. In words, at each step the random walk jumps to one of its $(n/2)+1$ nearest neighbors (including the point it's already on), each with equal probability.

Prove that there are constants $A, \alpha > 0$ (independent of n and k) such that

$$\|\mu_k - \pi\| \;\le\; Ae^{-\alpha k}\,, \qquad \text{for all } n, k\,.$$

In particular, this proves that this random walk converges in $O(1)$ steps, i.e. in a number of steps which does not depend on n.

Minorisation conditions

9. Let $\mathcal{X} = \{0, 1, 2, 3\}$, and let

$$P \;=\; \begin{pmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.4 & 0.2 & 0.4 & 0 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0.4 & 0 & 0.4 & 0.2 \end{pmatrix}.$$

Let $\mu_0(0) = 1$ give the initial distribution. Find an exponentially decreasing bound on the variation distance $\|\mu_k - \pi\|$. (Hint: Use a uniform minorisation condition!)

10. Consider a Markov chain with transition kernel $P(x, \cdot)$ on a state space $\mathcal{X} \subseteq \mathbf{R}$, with stationary distribution $\pi(\cdot)$. Suppose $P(x, dy) = f(x, y)\, dy$ for some non-negative Lebesgue-measurable function f.

(a) Find (with proof) the largest possible value of β so that

$$P(x, \cdot) \;\ge\; \beta\, Q(\cdot), \qquad x \in \mathcal{X}$$

for some probability distribution Q on $\mathcal{X}$.

(b) Use this to get a bound on $\|\mu_k - \pi\|_{\text{var}}$, where π is the (unique) stationary distribution for P.

11. *Multi-step minorisation conditions.* Suppose we are given that $P^{k_0}(x, \cdot) \ge \beta\zeta(\cdot)$ for all $x \in \mathcal{X}$, for some positive integer k_0. Prove that

$$\|\mu_k - \pi\| \;\le\; (1-\beta)^{[k/k_0]}\,,$$

where $[k/k_0]$ means the greatest integer not exceeding k/k_0. (Hint: First prove it for k an integer multiple of k_0, and then use exercise 2.)

Advanced questions

12. *Convergence of projections.* Let $\mathcal{X} = \mathbf{Z}/(80) \times \mathbf{Z}/(100)$ be the abelian group consisting of pairs (x, y) with $x \in \mathbf{Z}/(80)$ and $y \in \mathbf{Z}/(100)$, with addition defined coordinate-wise as usual. Let the step distribution $Q(\cdot)$ on $\mathcal{X}$ be defined by

$$Q(0,0) = 1/2; \qquad Q(1,1) = Q(-1,1) = Q(1,-1) = Q(-1,-1) = 1/8\,.$$

As usual, let $P\left((x_1, x_2), (y_1, y_2)\right) = Q(y_1 - x_1, y_2 - x_2)$, define μ_0 by $\mu_0(0,0) = 1$, and let $\mu_k = \mu_0 P^k$ be the distribution of the random walk after k steps.

Furthermore, let ν_k be the distribution of the first coordinate after k steps. Formally,

$$\nu_k(x_1) \;=\; \sum_{x_2 \in \mathbf{Z}/(100)} \mu_k(x_1, x_2)\,.$$

Let π be the uniform distribution on $\mathcal{X}$, and let π_1 be the uniform distribution on $\mathbf{Z}/(80)$. Prove or disprove each of the following assertions.

(a) $\|\mu_k - \pi\|_{\text{var}} \;\to\; 0 \quad \text{as} \quad k \to \infty.$

(b) $\|\nu_k - \pi_1\|_{\text{var}} \;\to\; 0 \quad \text{as} \quad k \to \infty.$

13. *Random walk on the chessboard, Part I.* Consider the group $\mathbf{Z}/(n) \times \mathbf{Z}/(m)$, thought of as an $n \times m$ rectangular grid, to be used as a "chessboard". (In this interpretation, we must allow the chesspieces to "wrap around", in the sense that a chesspiece at (say) the right edge of the board could jump off to the right and re-appear on the left.) For each of the following three step distributions, get upper and lower bounds on the distance to stationarity for the corresponding random walk. (For simplicity, in this exercise you may neglect lower-order terms as $n, m \to \infty$.)

(a) The factorable king moves: $Q(a, b) = 1/9$ for $a, b = -1, 0, 1$. (Hint: Write this random walk as a "product" of two simpler random walks, and use exercise 1.)

(b) The non-factorable king moves: $Q(0,0) = 1/2$, $Q(a,b) = 1/16$ for $a, b = -1, 0, 1$, with $(a, b) \neq (0, 0)$.

(c) The knight moves: $Q(\pm 1, \pm 2) = Q(\pm 2, \pm 1) = Q(0,0) = 1/9$.

14. *Random walk on a stick.* Let $\mathcal{X} = \{1, 2, \ldots, n\}$, thought of as n points in a line (not in a circle). Define the transition probabilities P by

$$P(x, x-1) = P(x, x) = P(x, x+1) = 1/3, \qquad 2 \le x \le n-1\,;$$

$$P(1,1) = P(n,n) = 1/3; \qquad P(1,2) = P(n, n-1) = 2/3\,.$$

In words, the Markov chain jumps left or right, or stays still, each with probability 1/3, *except at the endpoints.* At the endpoints, it stays still with probability 1/3 and jumps back towards the center with probability 2/3. (This corresponds to having "reflecting barriers" at the endpoints.)

You are to bound the convergence of this Markov chain by *lifting* it to a random walk on the group $\mathbf{Z}/(2n-2) = \{1', 2', 3', \ldots, (2n-2)'\}$, as follows. Identify the point 1 with the point $1'$, and the point n with the point n'. For $2 \leq x \leq n-1$, identify the point x with the point x' and with the point $(2n-x)'$.

(a) Argue that our "usual" random walk on $\mathbf{Z}/(2n-2)$ (which at each step moves left, right, or not at all, each with probability 1/3) "projects" (explain what that means!) under this identification onto the Markov chain on $\mathcal{X}$.

(b) Use this "projection" to determine the stationary distribution π of our Markov chain on $\mathcal{X}$.

(c) Argue that the variation distance to stationarity of the Markov chain on $\mathcal{X}$ is bounded above by the corresponding variation distance on the "covering chain" on $\mathbf{Z}/(2n-2)$.

(d) Use this to derive upper bounds on the rate of convergence of the Markov chain to the stationary distribution π.

15. *Random walk on the chessboard, Part II.* Redo the analysis of the non-factorable king moves on a chessboard (exercise 13(b)), but with the "wrap around" assumption replaced by reflecting barriers on all sides. In other words, if the king is at one edge of the chessboard, then the probabilities that would normally make it move off the edge are instead added on to the probability of moving to the "reflected" point the other way. (That is, the new probabilities are designed so they can "lift" to a random walk on $\mathbf{Z}/(2n-2) \times \mathbf{Z}/(2m-2)$, similar to the previous question.) For example, $P((0,2),(1,3)) = 2/16 = 1/8$, because the probability of moving to $(-1,3)$ is instead added on to the probability of moving to $(1,3)$. Also $P((0,0),(1,1)) = 4/16 = 1/4$.

For this new, "reflecting barriers" king on the $n \times m$ chessboard, get upper bounds on the rate of convergence to stationarity. (Once again, you may neglect lower-order terms as $n, m \to \infty$.)