STA 3047F, Fall 2001: Homework #2

Jeffrey S. Rosenthal, Department of Statistics, University of Toronto

Due in Sidney Smith Hall room 6024 by 4:00 p.m. on Monday November 19, 2001.

1. Prove that weak limits, if they exist, are *unique*. That is, if $\mu, \nu, \mu_1, \mu_2, \ldots$ are probability measures, and $\{\mu_n\} \Rightarrow \mu$, and also $\{\mu_n\} \Rightarrow \nu$, then $\mu = \nu$.

2. Let $f : [0,1] \to [0,\infty)$ be a continuous function such that $\int_0^1 f\, d\lambda = 1$ (where λ is Lebesgue measure on $\mathbf{R}$). Define probability measures μ and $\{\mu_n\}$ by $\mu(A) = \int f\, \mathbf{1}_A\, d\lambda$ and $\mu_n(A) = \sum_{i=1}^n f(i/n)\, \mathbf{1}_A(i/n) \;\; / \;\; \sum_{i=1}^n f(i/n)$.

(a) Prove that $\mu_n \Rightarrow \mu$.

(b) Explicitly construct random variables Y and $\{Y_n\}$ so that $\mathcal{L}(Y) = \mu$, $\mathcal{L}(Y_n) = \mu_n$, and $Y_n \to Y$ with probability 1.

3. Let $\mu_n = N(0, \frac{1}{n})$ be a normal distribution with mean 0 and variance $\frac{1}{n}$. Does the sequence $\{\mu_n\}$ converge weakly to some probability measure? If yes, to what measure?

4. Let $a_1, a_2, \ldots$ be any sequence of non-negative real numbers with $\sum_i a_i = 1$. Define the discrete measure μ by $\mu(\cdot) = \sum_{i \in \mathbf{N}} a_i \delta_i(\cdot)$, where $\delta_i(\cdot)$ is a point-mass at the positive integer i. Construct a sequence $\{\mu_n\}$ of *absolutely continuous* measures μ_n, such that $\mu_n \Rightarrow \mu$.

5. Let μ_n be the **Poisson**(n) distribution, and let μ be the **Poisson**(5) distribution. Show explicitly that each of the four equivalent definitions of weak convergence are violated.

6. Let $M > 0$, and let $f, f_1, f_2, \ldots : [0,1] \to [\frac{1}{M}, M]$ be Borel-measurable functions with $\int_0^1 f\, d\lambda = \int_0^1 f_n\, d\lambda = 1$. Suppose $\lim_{n\to\infty} \frac{f_n(x)}{f(x)} = 1$ for each fixed $x \in [0,1]$. Define probability measures $\mu, \mu_1, \mu_2, \ldots$ by $\mu(A) = \int_A f\, d\lambda$ and $\mu_n(A) = \int_A f_n\, d\lambda$, for Borel $A \subseteq [0,1]$. Prove that $\mu_n \Rightarrow \mu$.