STA 3047F, Fall 2001: Homework #3

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Due in Sidney Smith Hall room 6024 by 3:00 p.m. on Friday December 7, 2001.

1. Let $\mathcal{X}$ be a Polish space. Let $C \subseteq \mathcal{X}$ be a subset of $\mathcal{X}$, with the same metric.

(a) Provide a counter-example to show that C might not be a Polish space.

(b) Prove that if C is a *closed* subset of $\mathcal{X}$, then C must be a Polish space.

2. Suppose each of P_1 and P_2 are reversible with respect to $\pi(\cdot)$. For each of the following statements, either prove it is always true, or provide a counter-example to show it is sometimes false.

(a) $P_1 P_2$ is reversible with respect to π, where $(P_1 P_2)(x, A) = \int P_1(x, dy) P_2(y, A)$.

(b) $\frac{1}{2}(P_1 + P_2)$ is reversible with respect to π, where $\frac{1}{2}(P_1 + P_2)(x, A) = \frac{1}{2}P_1(x, A) + \frac{1}{2}P_2(x, A)$.

(c) P_1^2 is reversible with respect to π, where $(P_1^2)(x, A) = \int P_1(x, dy) P_1(y, A)$.

(d) $\frac{1}{2}(P_1^2 + P_2)$ is reversible with respect to π, where $\frac{1}{2}(P_1^2 + P_2)(x, A) = \frac{1}{2}P_1^2(x, A) + \frac{1}{2}P_2(x, A)$.

3. Let $\mathcal{X} = \mathbf{R}^d$, and let λ be d-dimensional Lebesgue measure. Let f be a density with respect to λ. For $x \in \mathcal{X}$ and $i \in \{1, 2, \ldots, d\}$, let $H_{x,i} = \{y \in \mathcal{X};\ y_j = x_j \text{ for } j \neq i\}$. (That is, $H_{x,i}$ is the line through x which is parallel to the i^{th} coordinate axis.) Let $q_i(x, y) = 0$ if $y \notin H_{x,i}$, and $q_i(x, y) = f(y) \,/\, \int_{z \in H_{x,i}} f(z) \lambda(dz)$ if $y \in H_{x,i}$. Fix $i \in \{1, 2, \ldots, d\}$. Consider running a Metropolis-Hastings algorithm with target distribution $f(x)\lambda(dx)$, and with proposal distribution $q_i(x, y)\lambda(dy)$. Derive a simple formula for the acceptance probability $\alpha(x, y)$. [The resulting algorithm is one sub-step of the *Gibbs sampler*.]