1. Let $\mathcal{X}$ be a Polish space. Let $C \subseteq \mathcal{X}$ be a subset of $\mathcal{X}$, with the same metric.

(a) Provide a counter-example to show that $C$ might not be a Polish space.

(b) Prove that if $C$ is a closed subset of $\mathcal{X}$, then $C$ must be a Polish space.

2. Suppose each of $P_1$ and $P_2$ are reversible with respect to $\pi(\cdot)$. For each of the following statements, either prove it is always true, or provide a counter-example to show it is sometimes false.

(a) $P_1 P_2$ is reversible with respect to $\pi$, where $(P_1 P_2)(x, A) = \int P_1(x, dy) P_2(y, A)$.

(b) $\frac{1}{2} (P_1 + P_2)$ is reversible with respect to $\pi$, where $\frac{1}{2} (P_1 + P_2)(x, A) = \frac{1}{2} P_1(x, A) + \frac{1}{2} P_2(x, A)$.

(c) $P_i^2$ is reversible with respect to $\pi$, where $P_i^2(x, A) = \int P_1(x, dy) P_i(y, A)$.

(d) $\frac{1}{2} (P_i^2 + P_2)$ is reversible with respect to $\pi$, where $\frac{1}{2} (P_i^2 + P_2)(x, A) = \frac{1}{2} P_i^2(x, A) + \frac{1}{2} P_2(x, A)$.

3. Let $\mathcal{X} = \mathbb{R}^d$, and let $\lambda$ be $d$-dimensional Lebesgue measure. Let $f$ be a density with respect to $\lambda$. For $x \in \mathcal{X}$ and $i \in \{1, 2, \ldots, d\}$, let $H_{x,i} = \{y \in \mathcal{X}; y_j = x_j \text{ for } j \neq i\}$. (That is, $H_{x,i}$ is the line through $x$ which is parallel to the $i^{th}$ coordinate axis.) Let $q_i(x, y) = 0$ if $y \notin H_{x,i}$, and $q_i(x, y) = f(y) / \int_{z \in H_{x,i}} f(z) \lambda(dz)$ if $y \in H_{x,i}$. Fix $i \in \{1, 2, \ldots, d\}$. Consider running a Metropolis-Hastings algorithm with target distribution $f(x) \lambda(dx)$, and with proposal distribution $q_i(x, y) \lambda(dy)$. Derive a simple formula for the acceptance probability $\alpha(x, y)$. [The resulting algorithm is one sub-step of the Gibbs sampler.]