UNIVERSITY OF TORONTO
Faculty of Arts and Science
APRIL/MAY EXAMINATIONS 2001
STA447H1 S

[Also cross-listed as STA2006H.]

Duration – 3 hours

NO AIDS ALLOWED.

(Number of questions: 11. Number of pages: 3. Total number of points: 140.)

1. (10 points) Consider a (discrete-time) Markov chain on a finite state space S, which is not irreducible, and such that every state has period exactly 3. Either provide (with explanation) an example of such a chain, or prove that no such chain exists.

2. (10 points) Consider a (discrete-time) Markov chain on the state space $S = \{1, 2, 3\}$, such that $p_{13}^{(n)} < 1$ for all $n \in \mathbf{N}$, but $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$. Either provide (with explanation) an example of such a chain, or prove that no such chain exists.

3. (10 points) Consider a (discrete-time) Markov chain on the state space $S = \{1, 2\}$, such that $p_{12} > 0$, and $p_{21} > 0$, and $f_{12} = 1/3$. Either provide (with explanation) an example of such a chain, or prove that no such chain exists.

4. (10 points) Consider a (discrete-time) Markov chain on the state space $S = \{1, 2, 3, 4, 5\}$, with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute (with explanation) the value of f_{25}. [Hint: Don't forget the Gambler's Ruin problem.]

5. (20 points) Consider the Ehrenfest's Urn Markov chain $\{X_n\}$ with $d = 3$, so that $S = \{0, 1, 2, 3\}$ and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\pi = (1/8,\ 3/8,\ 3/8,\ 1/8)$.

(a) Verify explicitly that π is a stationary distribution for P.

(b) Prove or disprove that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$.

(c) Compute (with explanation) the mean return time to the state 0. [Hint: This doesn't require much computation.]

(d) Let $N(t) = \#\{n \le t\,;\ X_n = 0\}$ be the number of times the chain has hit the state 0 by time t. Compute (with explanation) the value of $\lim_{t\to\infty} N(t)\,/\,t$.

6. (10 points) Let $N \ge 3$ be an integer. Consider a (discrete-time) Markov chain on the state space $S = \{1, 2, \ldots, N\}$. Suppose $p_{ij} = 0$ whenever $|i - j| \ge 2$. Suppose also that $p_{i,i+1} = 2\, p_{i+1,i}$ for $1 \le i \le N-1$. Find a distribution $\{\pi_i\}_{i\in S}$ which must be a stationary distribution for the chain. [Hint: Use time reversibility.]

7. (15 points) Consider a Markov chain $\{X_n\}$ on $S = \{1, 2, 3, \ldots, 101\}$, with transition probabilities defined as follows. For $j \ge 11$, $p_{j,1} = p_{j,2} = p_{j,3} = 1/3$. For $j \le 10$, $p_{j,j^2+1} = 1$.

(a) (5 points) Let $N(t) = \#\{n \le t\,;\ X_n \ge 11\}$ be the number of times the chain has exceeded the value 10 by time t. Is $\{N(t)\}$ a renewal process? (Explain your answer.)

(b) (10 points) Compute

$$\lim_{n\to\infty} \mathbf{P}(5 \le X_n \le 9)\,.$$

[Hint: Use W.L. Smith's Theorem.]

8. (15 points) Let $\{X_n\}$ be a discrete-time Markov chain on $S = \{0, 1, 2, 3\}$, with transition probability matrix

$$P \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $X_0 = 0$, let $T_0 = 0$, and for $i \ge 1$ let $T_i = \min\{m > T_{i-1}\,;\ X_m = 0\}$ be the i^{th} return of $\{X_n\}$ to the state 0. Let

$$R_n = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \le n\}}\, (T_i - T_{i-1})^2\,.$$

Compute $\lim_{n\to\infty} R_n\,/\,n$. [Hint: Is $\{R_n\}$ a Renewal Reward Process?]

9. (10 points) Let $A \ge 1$ and $B \ge 2$ be integers. Consider the Markov chain $\{X_n\}$ on $\{-A, -A+1, \ldots, -1, 0, 1, \ldots, B-1, B\}$, defined by $X_0 = 0$, and $p_{-A,-A} = p_{B,B} = 1$, and $p_{B-1,B} = p_{B-1,B-2} = \frac{1}{2}$, and $p_{i,i+2} = 1/3$ and $p_{i,i-1} = 2/3$ for $-A+1 \le i \le B-2$. Let $T = \min\{n \ge 1\,;\ X_n = -A \text{ or } X_n = B\}$. Compute (with explanation) the value of $\mathbf{P}[X_T = B]$. [Hint: Don't forget about martingales.]

10. (20 points) Let a and M be integers, with $0 \leq a \leq M-2$. Let $\{X_n\}$ be a Markov chain on the state space $S = \{-M, -M+1, \ldots, -1, 0, 1, \ldots, M\}$, with $X_0 = a$, and transition probabilities defined by: $p_{0,1} = p_{0,0} = p_{0,-1} = \frac{1}{3}$, and $p_{-M,M} = p_{M,-M} = 1$, and for $i \in S$ with $1 \leq |i| \leq M-1$, $p_{i,-i+1} = p_{i,-i-1} = \frac{1}{2}$. Let $Y_n = (-1)^n X_n$.

(a) (5 points) Prove that $\{Y_n\}$ is a martingale.

(b) (5 points) Compute $\mathbf{E}[X_n]$, for $n \in \mathbf{N}$. [Hint: $X_n = (-1)^n Y_n$.]

(c) (10 points) Let $T = \min\{n \geq 1\,;\; X_n = M\}$. Compute $\mathbf{P}[T \text{ is an odd number}]$. [Hint: Consider Y_T.]

11. (10 points) Let $\{B(t)\}_{t \geq 0}$ be standard Brownian motion, with $B(0) = 0$.

(a) For $n \in \mathbf{N}$, compute (with explanation) $\mathbf{E}[(B_n - B_{n-1})^2]$.

(b) Compute (with explanation) $\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} (B_n - B_{n-1})^2$. [Hint: Use the Strong Law of Large Numbers.]

END OF EXAM. TOTAL MARKS = 140.