STA 447/2006S, Spring 2001: Homework #1

Due by Monday, February 12, 4:00 p.m., in Sid Smith 6024.

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Copying other solutions is strictly prohibited!

All questions are from the book Probability and Random Processes, Second Edition, by G.R. Grimmett and D.R. Stirzaker (Oxford University Press, 1992, available for purchase at http://www.oupcan.com/index.shtml). I have re-typed the questions here – see note at end.

Page 200: Section 6.1, Exercise 17 [10 points] Let X_n be the maximum reading obtained in the first n throws of a fair die. Show that X is a Markov chain, and find the transition probabilities $p_{ij}(n)$.

Page 201: Section 6.1, Exercise 22 [15 points] Let X be a Markov chain with state space S, and suppose that $h : S \to T$ is one-one. Show that $Y_n = h(X_n)$ defines a Markov chain on T. Must this be so if h is not one-one?

Page 207: Section 6.3, Exercise 10(a) [10 points] A particle performs a random walk on the vertices of a cube. At each step if remains where it is with probability $\frac{1}{4}$, and moves to each of its neighbouring vertices with probability $\frac{1}{4}$. If the walk starts at a vertex v, find the mean number of steps until its first return to v.

Page 218: Section 6.4, Exercise 27 [10 points] Show by example that chains which are not irreducible may have many different stationary distributions.

Page 238: Section 6.8, Exercise 26 [10 points] **Superposition.** Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities λ and μ. Show that the arrivals of flying objects form a Poisson process with intensity $\lambda + \mu$.

Page 264: Section 6.13, Exercise 1 [20 points] Classify the states of the discrete-time Markov chains with state space $S = \{1, 2, 3, 4\}$ and transition matrices

$$\text{(a)} \quad \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Page 264: Section 6.13, Exercise 2 [20 points] A transition matrix is called *doubly stochastic* if all its column sums equal 1: that is, if $\sum_i p_{ij} = 1$ for all $j \in S$.

(a) Show that if a finite chain has a doubly stochastic transition matrix, then all its states are non-null persistent, and that if it is, in addition, irreducible and aperiodic then $p_{ij}(n) \to N^{-1}$ as $n \to \infty$, where N is the number of states.

(b) Show that, if an infinite irreducible chain has a doubly stochastic transition matrix, then its states are either all null persistent or all transient.

Page 264: Section 6.13, Exercise 4 [15 points for parts (b) and (c) assuming the result of part (a), plus 10 bonus points if you also get part (a) which is more difficult]

(a) Show that for each pair i, j of states of an irreducible aperiodic chain, there exists $N = N(i, j)$ such that $p_{ij}(n) > 0$ for all $n \geq N$.

(b) Let X and Y be independent irreducible aperiodic Markov chains with the same state space S and same transition matrix P. Show that the bivariate chain $Z_n = (X_n, Y_n)$, $n \geq 0$, is irreducible and aperiodic.

(c) Show that the bivariate chain Z may be reducible if X and Y are periodic.

Page 265: Section 6.13, Exercise 9 [15 points] Consider the symmetric random walk in three dimensions on the set of points $\{(x, y, z) : x, y, z = 0, \pm 1, \pm 2, \ldots\}$; this process is a sequence $\{\mathbf{X}_n : n \geq 0\}$ of points such that $\mathbf{P}(\mathbf{X}_{n+1} = \mathbf{X}_n + \epsilon) = 1/6$ for $\epsilon = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$. Suppose that $\mathbf{X}_0 = (0, 0, 0)$. Show that

$$\mathbf{P}\Big(\mathbf{X}_{2n} = (0,0,0)\Big) = (1/6)^{2n} \sum_{i+j+k=n} \frac{(2n)!}{(i!j!k!)^2}$$

$$= (1/2)^{2n} \binom{2n}{n} \sum_{i+j+k=n} \left(\frac{n!}{(3^n\, i!j!k!)^2} \right)^2$$

and deduce by Stirling's formula that the origin is a transient state.

A note about the textbook

It has come to my attention that some students are hesitant to purchase the textbook for this course (*Probability and random processes, second edition*, by G.R. Grimmett and D.R. Stirzaker, Oxford University Press, 1992, available for purchase at http://www.oupcan.com/index.shtml).

I do strongly recommend that you obtain some book from which you can study the course material. This is because, on the tests and exams, you will be expected to understand the material more completely than just the outline I provide in lectures (for example, you should be able to work out examples and prove theorems which are different from what I've done in lecture).

If you don't like the course textbook, other books which cover large portions of the course material include:

S. Asmussen (1987), Applied probability and queues. John Wiley & Sons, New York.

W. Feller (1968), An introduction to probability theory and its applications, Vol. I (3rd ed.). Wiley & Sons, New York.

S. Karlin and H. M. Taylor (1981), A second course in stochastic processes. Academic Press, New York. [They also have a "first course in stochastic processes" which might be easier but I'm not familiar with it.]

S. Resnick (1992), Adventures in stochastic processes. Birkhäuser, Boston.

As a courtesy to those students who elect not to purchase the course textbook, I have decided to re-type the relevant homework questions here. However, please note that despite my doing this, I still strongly encourage you to purchase and study some book which discusses the course material.