1. (10 points) Consider a (discrete-time) Markov chain on the state space \( S = \{1, 2, 3\} \) such that
\[
P = \begin{pmatrix}
\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\
\frac{1}{3} & 0 & \frac{2}{3} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}.
\]
Suppose that \( \mu_1^{(0)} = 1 \), and \( \mu_2^{(0)} = \mu_3^{(0)} = 0 \). Compute \( \mu_1^{(2)} \). (Explain your reasoning.)

Solution. We compute that
\[
\mu_1^{(2)} = \sum_j \sum_k \mu_j^{(0)} p_{jk} p_{k1} = \sum_k p_{1k} p_{k1}
\]
\[
= p_{11} p_{11} + p_{12} p_{21} + p_{13} p_{31} = \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{2}\right) = \frac{28}{75}.
\]

2. (10 points) Give (with explanation) an example of a valid transition matrix \( P \) for a (discrete-time) Markov chain on the state space \( S = \{1, 2, 3\} \), with the property that
\[
0 < f_{12} < 1.
\]

Solution. It suffice to have \( p_{12} > 0 \), and \( p_{13} > 0 \), and \( p_{33} = 1 \). That way, \( f_{12} \geq p_{12} > 0 \). Also, if the chain hits 3 right away then it will never leave 3 and hence never hit 2, so that \( f_{12} \leq 1 - p_{13} < 1 \). Specific examples for \( P \) include
\[
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 1
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0.8 & 0.1 & 0.1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

3. (10 points) Let \( \{N(t)\}_{t \geq 0} \) be a Poisson process with constant intensity \( \lambda > 0 \). Compute (with explanation) the conditional probability
\[
P(N(2.6) = 2 \mid N(2.9) = 2).
\]

Solution. We compute that
\[
P(N(2.6) = 2 \mid N(2.9) = 2) = \frac{P(N(2.6) = 2, N(2.9) = 2)}{P(N(2.9) = 2)}
\]
\[
\frac{P(N(2.6) = 2, N(2.9) - N(2.6) = 0)}{P(N(2.9) = 2)} = \frac{P(N(2.6) = 2) \cdot P(N(2.9) - N(2.6) = 0)}{P(N(2.9) = 2)} \quad \text{(since indep. increments)}
\]
\[
= \frac{(e^{-2.6\lambda}(2.6\lambda)^2/2!) \cdot (e^{-0.3\lambda}(0.3\lambda)^0/0!)}{(e^{-2.9\lambda}(2.9\lambda)^2/2!)} \quad \text{(since Poisson)}
\]
\[
= \frac{(2.6/2.9)^2.}
\]

**Remark.** This answer does not depend on \(\lambda\). Can you explain why not?

4. (15 points) Consider a (discrete-time) Markov chain on the state space \(S = \{1, 2, 3, 4\}\) such that
\[
P = \begin{pmatrix}
1/2 & 1/2 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 \\
0 & 0 & 1/2 & 1/2
\end{pmatrix}.
\]
Prove or disprove the following assertion:
\[
\lim_{n \to \infty} p_{ij}(n) = 1/4, \quad \text{for all } i,j \in S.
\]

**Solution.** Firstly, the chain is irreducible. Indeed, \(p_{12} > 0, p_{23} > 0, p_{34} > 0\).
Also \(p_{13}(2) \geq p_{12}p_{23} > 0, p_{24}(2) \geq p_{23}p_{34} > 0\). And \(p_{14}(3) \geq p_{12}p_{23}p_{34} > 0\).
Similarly, \(p_{21} > 0, p_{32} > 0, p_{43} > 0\). Also \(p_{31}(2) \geq p_{32}p_{21} > 0, p_{42}(2) \geq p_{43}p_{32} > 0\).
And \(p_{41}(3) \geq p_{43}p_{32}p_{21} > 0\).

Secondly, the chain is aperiodic. Indeed, \(p_{11} > 0\), so state 1 is aperiodic. But by a theorem in class, for an irreducible Markov chain, if one state is aperiodic then all states are aperiodic.

Thirdly, \(\pi = (1/4, 1/4, 1/4, 1/4)\) is a stationary distribution for the chain. Indeed, since the matrix is symmetric (i.e. \(p_{ij} = p_{ji}\) for all \(i, j \in S\)), therefore \((1/4)p_{ij} = (1/4)p_{ji}\) for all \(i, j \in S\), so that the chain is time-reversible with respect to \(\pi\). [Or, compute directly that \(\pi P = \pi\). Or, use a result from the last homework about doubly-stochastic matrices.]

Finally, from the “limit theorem” in class, for an irreducible, aperiodic Markov chain with transition probabilities \(\{p_{ij}\}\) and stationary distribution \(\pi\), we know that \(\lim_{n \to \infty} p_{ij}(n) = \pi_j\) for all \(i, j \in S\). Hence, the statement is true and proved.