# STA 447/2006S, Spring 2001, Test #1: SOLUTIONS

**1.** (10 points) Consider a (discrete-time) Markov chain on the state space $S = \{1, 2, 3\}$ such that

$$P = \begin{pmatrix} 1/5 & 2/5 & 2/5 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

Suppose that $\mu_1^{(0)} = 1$, and $\mu_2^{(0)} = \mu_3^{(0)} = 0$. Compute $\mu_1^{(2)}$. (Explain your reasoning.)

**Solution.** *We compute that*

$$\mu_1^{(2)} = \sum_j \sum_k \mu_j^{(0)} p_{jk} p_{k1} = \sum_k p_{1k} p_{k1}$$

$$= p_{11}p_{11} + p_{12}p_{21} + p_{13}p_{31} = (1/5)(1/5) + (2/5)(1/3) + (2/5)(1/2) = 28/75\,.$$

**2.** (10 points) Give (with explanation) an example of a valid transition matrix $P$ for a (discrete-time) Markov chain on the state space $S = \{1, 2, 3\}$, with the property that

$$0 < f_{12} < 1\,.$$

**Solution.** *It suffice to have $p_{12} > 0$, and $p_{13} > 0$, and $p_{33} = 1$. That way, $f_{12} \geq p_{12} > 0$. Also, if the chain hits 3 right away then it will never leave 3 and hence never hit 2, so that $f_{12} \leq 1 - p_{13} < 1$. Specific examples for $P$ include*

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**3.** (10 points) Let $\{N(t)\}_{t \geq 0}$ be a Poisson process with constant intensity $\lambda > 0$. Compute (with explanation) the conditional probability

$$\mathbf{P}\big(N(2.6) = 2 \;\big|\; N(2.9) = 2\big)\,.$$

**Solution.** *We compute that*

$$\mathbf{P}\big(N(2.6) = 2 \;\big|\; N(2.9) = 2\big) = \frac{\mathbf{P}\big(N(2.6) = 2\,,\; N(2.9) = 2\big)}{\mathbf{P}\big(N(2.9) = 2\big)}$$

$$= \frac{\mathbf{P}\big(N(2.6) = 2\,,\ N(2.9) - N(2.6) = 0\big)}{\mathbf{P}\big(N(2.9) = 2\big)}$$

$$= \frac{\mathbf{P}\big(N(2.6) = 2\big)\ \mathbf{P}\big(N(2.9) - N(2.6) = 0\big)}{\mathbf{P}\big(N(2.9) = 2\big)} \qquad \text{(since indep. increments)}$$

$$= \frac{(e^{-2.6\lambda}(2.6\lambda)^2\,/\,2!)\ (e^{-0.3\lambda}(0.3\lambda)^0\,/\,0!)}{(e^{-2.9\lambda}(2.9\lambda)^2\,/\,2!)} \qquad \text{(since Poisson)}$$

$$= \ (2.6\,/\,2.9)^2\,.$$

**Remark.** This answer does not depend on $\lambda$. Can you explain why not?

**4.** (15 points) Consider a (discrete-time) Markov chain on the state space $S = \{1, 2, 3, 4\}$ such that

$$P \ = \ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Prove or disprove the following assertion:

$$\lim_{n\to\infty} p_{ij}(n) = 1/4\,, \qquad \text{for all } i, j \in S\,.$$

**Solution.** *Firstly, the chain is irreducible. Indeed,* $p_{12} > 0$, $p_{23} > 0$, $p_{34} > 0$. *Also* $p_{13}(2) \geq p_{12}p_{23} > 0$, $p_{24}(2) \geq p_{23}p_{34} > 0$. *And* $p_{14}(3) \geq p_{12}p_{23}p_{34} > 0$. *Similarly,* $p_{21} > 0$, $p_{32} > 0$, $p_{43} > 0$. *Also* $p_{31}(2) \geq p_{32}p_{21} > 0$, $p_{42}(2) \geq p_{43}p_{32} > 0$. *And* $p_{41}(3) \geq p_{43}p_{32}p_{21} > 0$.

*Secondly, the chain is aperiodic. Indeed,* $p_{11} > 0$, *so state 1 is aperiodic. But by a theorem in class, for an irreducible Markov chain, if one state is aperiodic then all states are aperiodic.*

*Thirdly,* $\pi = (1/4,\ 1/4,\ 1/4,\ 1/4)$ *is a stationary distribution for the chain. Indeed, since the matrix is symmetric (i.e.* $p_{ij} = p_{ji}$ *for all* $i, j \in S$*), therefore* $(1/4)p_{ij} = (1/4)p_{ji}$ *for all* $i, j \in S$*, so that the chain is time-reversible with respect to* $\pi$*. [Or, compute directly that* $\pi P = \pi$*. Or, use a result from the last homework about doubly-stochastic matrices.]*

*Finally, from the "limit theorem" in class, for an irreducible, aperiodic Markov chain with transition probabilities* $\{p_{ij}\}$ *and stationary distribution* $\pi$*, we know that* $\lim_{n\to\infty} p_{ij}(n) = \pi_j$ *for all* $i, j \in S$*. Hence, the statement is true and proved.*