STA 447/2006S, Spring 2001, Test #2: SOLUTIONS

1. (10 points) Consider a single-server queue with interarrival time distribution $\mathbf{Exp}(\lambda)$, and service time distribution $\mathbf{Unif}[0,10]$. Let W_n be the waiting time of the n^{th} customer. Give (with explanation) necessary and sufficient conditions on λ such that $W_n \to \infty$ in probability.

> **Solution.** *Here the mean interarrival time is* $1/\lambda$, *and the mean service time is* 5. *Hence, the traffic density is* $\rho = 5/(1/\lambda) = 5\lambda$. *Now, we know from class that* $W_n \to \infty$ *in probability if and only if* $\rho \geq 1$, *i.e. if and only if* $5\lambda \geq 1$, *or* $\lambda \geq 0.2$.

2. (10 points) Let $\{N(t)\}$ be a non-arithmetic renewal process with finite mean interarrival time μ. Fix $h > 0$. Compute (with explanation) the limit

$$\lim_{t\to\infty}\left(\frac{N(t+h)-N(t)}{t}\right)^2 .$$

> **Solution.** *From the first part of the Elementary Renewal Theorem, we know that as* $t \to \infty$, *with probability 1,* $N(t)/t \to 1/\mu$. *Hence, with probability 1,* $N(t+h)/t = (N(t+h)/(t+h))((t+h)/t) \to (1/\mu)(1) = 1/\mu$. *Thus, with probability 1,* $(N(t+h)-N(t))/t \to (1/\mu)-(1/\mu) = 0$, *so also* $((N(t+h)-N(t))/t)^2 \to 0^2 = 0$ *with probability 1. Hence,* $\lim_{t\to\infty}((N(t+h)-N(t))/t)^2 = 0$.

3. (15 points) Let a and c be positive integers, with $0 < a < c-1$. Consider the Gambler's Ruin Markov chain $\{X_n\}$ on $\{0,1,\ldots,c\}$ with $p = \frac{1}{2}$, so that $X_0 = a$, and $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$ for $1 \leq i \leq c-1$ and $p_{00} = p_{cc} = 1$. Define the stopping time U by $U = \min\{n \geq 1\,;\ X_n = a+1\}$.

(a) Show that $\{X_n\}$ is a martingale.

> **Solution.** *Clearly* $E|X_n| \leq c < \infty$ *for all* n. *Also, since* $\{X_n\}$ *is a Markov chain,* $\mathbf{E}(X_{n+1}\,|\,X_0,\ldots,X_n) = \mathbf{E}(X_{n+1}\,|\,X_n)$. *Now,* $\mathbf{E}(X_{n+1}\,|\,X_n = 0) = (1)(0) = 0$. *Also* $\mathbf{E}(X_{n+1}\,|\,X_n = c) = (1)(c) = c$. *If* $1 \leq i \leq c-1$ *then* $\mathbf{E}(X_{n+1}\,|\,X_n = i) = (\frac{1}{2})(i+1) + (\frac{1}{2})(i-1) = i$. *Hence, in any case,* $\mathbf{E}(X_{n+1}\,|\,X_0,\ldots,X_n) = \mathbf{E}(X_{n+1}\,|\,X_n) = X_n$.

(b) Prove or disprove that $\mathbf{E}[X_U] = \mathbf{E}[X_0]$.

> **Solution.** *If* $U < \infty$, *then* $X_U = a+1$ *by definition. However, with probability* $1/(a+1) > 0$, *the chain will stop at 0 before ever hitting* $a+1$, *in which case* $U = \infty$ *and* X_U *is not even well-defined. In either case, we do <u>not</u> have* $\mathbf{E}[X_U] = a$, *even though* $\mathbf{E}[X_0] = a$.

(c) Can the Optional Stopping Theorem (or its Corollary) be applied to this process

$\{X_n\}$ and stopping time U? (Explain your answer.)

Solution. *No, we cannot apply the Optional Stopping Theorem (or its Corollary) since we do not have* $\mathbf{P}(U < \infty) = 1$. *In fact,* $\mathbf{P}(U = \infty) = 1/(a+1) > 0$.

4. (15 points) Consider simple symmetric random walk $\{X_n\}$ on the set of all integers $\mathbf{Z}$, with $X_0 = 0$. Let $T_2 = \min\{n \geq 1\,;\ X_n = 2\}$. Prove or disprove that

$$\lim_{M\to\infty} \mathbf{E}[X_M \,|\, T_2 > M] \;=\; -\infty\,.$$

[Hint: You may wish to set $S = \min(T_2, M)$ and use the Law of Total Probability.]

Solution. *Let* $S = \min(T_2, M)$. *Then* S *is a stopping time with* $S \leq M$, *so by the Optional Sampling Theorem,* $\mathbf{E}[X_S] = \mathbf{E}[X_0] = 0$. *On the other hand, by the Law of Total Probability,*

$$\mathbf{E}[X_S] = \mathbf{P}(T_2 > M)\,\mathbf{E}[X_S \,|\, T_2 > M] + \mathbf{P}(T_2 \leq M)\,\mathbf{E}[X_S \,|\, T_2 \leq M]$$

$$= \mathbf{P}(T_2 > M)\,\mathbf{E}[X_M \,|\, T_2 > M] + [1 - \mathbf{P}(T_2 > M)]\,(2)$$

(since $S = M$ *whenever* $T_2 > M$, *while* $X_S = X_{T_2} = 2$ *whenever* $T_2 \leq M$*). Since* $\mathbf{E}[X_S] = 0$, *this says that*

$$\mathbf{E}[X_M \,|\, T_2 > M] \;=\; \frac{-[1 - \mathbf{P}(T_2 > M)]\,(2)}{\mathbf{P}(T_2 > M)}\,.$$

But simple symmetric random walk is recurrent, so that $\mathbf{P}(T_2 < \infty) = 1$, *and therefore* $\lim_{M\to\infty} \mathbf{P}(T_2 > M) = 0$. *Hence,*

$$\lim_{M\to\infty} \mathbf{E}[X_M \,|\, T_2 > M] \;=\; \lim_{M\to\infty} \frac{-[1 - \mathbf{P}(T_2 > M)]\,(2)}{\mathbf{P}(T_2 > M)} \;=\; -\infty\,.$$

Hence, the statement is true and proved.