1. (10 points) Consider a single-server queue with interarrival time distribution \( \text{Exp}(\lambda) \), and service time distribution \( \text{Unif}[0,10] \). Let \( W_n \) be the waiting time of the \( n^{\text{th}} \) customer. Give (with explanation) necessary and sufficient conditions on \( \lambda \) such that \( W_n \to \infty \) in probability.

Solution. Here the mean interarrival time is \( 1/\lambda \), and the mean service time is \( 5 \). Hence, the traffic density is \( \rho = 5/(1/\lambda) = 5\lambda \). Now, we know from class that \( W_n \to \infty \) in probability if and only if \( \rho \geq 1 \), i.e. if and only if \( 5\lambda \geq 1 \), or \( \lambda \geq 0.2 \).

2. (10 points) Let \( \{N(t)\} \) be a non-arithmetic renewal process with finite mean interarrival time \( \mu \). Fix \( h > 0 \). Compute (with explanation) the limit
\[
\lim_{t \to \infty} \left( \frac{N(t+h) - N(t)}{t} \right)^2.
\]

Solution. From the first part of the Elementary Renewal Theorem, we know that as \( t \to \infty \), with probability 1, \( N(t)/t \to 1/\mu \). Hence, with probability 1, \( N(t+h)/t = (N(t+h)/(t+h))(t+h)/t \to (1/\mu)(1) = 1/\mu \). Thus, with probability 1, \( (N(t+h) - N(t))/t \to (1/\mu) - (1/\mu) = 0 \), so also \( ((N(t+h) - N(t))/t)^2 \to 0^2 = 0 \) with probability 1. Hence, \( \lim_{t \to \infty} ((N(t+h) - N(t))/t)^2 = 0 \).

3. (15 points) Let \( a \) and \( c \) be positive integers, with \( 0 < a < c - 1 \). Consider the Gambler’s Ruin Markov chain \( \{X_n\} \) on \( \{0,1,\ldots,c\} \) with \( p = \frac{1}{2} \), so that \( X_0 = a \), and \( p_{i,i+1} = p_{i,i-1} = \frac{1}{2} \) for \( 1 \leq i \leq c - 1 \) and \( p_{00} = p_{cc} = 1 \). Define the stopping time \( U \) by \( U = \min\{n \geq 1; X_n = a + 1\} \).
   
   (a) Show that \( \{X_n\} \) is a martingale.

Solution. Clearly \( E[X_n] \leq c < \infty \) for all \( n \). Also, since \( \{X_n\} \) is a Markov chain, \( E(X_{n+1} \mid X_0, \ldots, X_n) = E(X_{n+1} \mid X_n) \). Now, \( E(X_{n+1} \mid X_n = 0) = (1)(0) = 0 \). Also \( E(X_{n+1} \mid X_n = c) = (1)(c) = c \). If \( 1 \leq i \leq c - 1 \) then \( E(X_{n+1} \mid X_n = i) = \left( \frac{1}{2} \right)(i+1) + \left( \frac{1}{2} \right)(i-1) = i \). Hence, in any case, \( E(X_{n+1} \mid X_0, \ldots, X_n) = E(X_{n+1} \mid X_n) = X_n \).

(b) Prove or disprove that \( E[X_U] = E[X_0] \).

Solution. If \( U < \infty \), then \( X_U = a + 1 \) by definition. However, with probability \( 1/(a+1) > 0 \), the chain will stop at 0 before ever hitting \( a + 1 \), in which case \( U = \infty \) and \( X_U \) is not even well-defined. In either case, we do not have \( E[X_U] = a \), even though \( E[X_0] = a \).

(c) Can the Optional Stopping Theorem (or its Corollary) be applied to this process?
\{X_n\} and stopping time \( U \)? (Explain your answer.)

**Solution.** No, we cannot apply the Optional Stopping Theorem (or its Corollary) since we do not have \( \mathbb{P}(U < \infty) = 1 \). In fact, \( \mathbb{P}(U = \infty) = 1/(a + 1) > 0 \).

4. (15 points) Consider simple symmetric random walk \( \{X_n\} \) on the set of all integers \( \mathbb{Z} \), with \( X_0 = 0 \). Let \( T_2 = \min\{n \geq 1; X_n = 2\} \). Prove or disprove that

\[
\lim_{M \to \infty} \mathbf{E}[X_M | T_2 > M] = -\infty.
\]

[Hint: You may wish to set \( S = \min(T_2, M) \) and use the Law of Total Probability.]

**Solution.** Let \( S = \min(T_2, M) \). Then \( S \) is a stopping time with \( S \leq M \), so by the Optional Sampling Theorem, \( \mathbf{E}[X_S] = \mathbf{E}[X_0] = 0 \). On the other hand, by the Law of Total Probability,

\[
\mathbf{E}[X_S] = \mathbf{P}(T_2 > M) \mathbf{E}[X_S | T_2 > M] + \mathbf{P}(T_2 \leq M) \mathbf{E}[X_S | T_2 \leq M]
\]

\[
= \mathbf{P}(T_2 > M) \mathbf{E}[X_M | T_2 > M] + [1 - \mathbf{P}(T_2 > M)] (2)
\]

(since \( S = M \) whenever \( T_2 > M \), while \( X_S = X_{T_2} = 2 \) whenever \( T_2 \leq M \)). Since \( \mathbf{E}[X_S] = 0 \), this says that

\[
\mathbf{E}[X_M | T_2 > M] = \frac{-[1 - \mathbf{P}(T_2 > M)] (2)}{\mathbf{P}(T_2 > M)}.
\]

But simple symmetric random walk is recurrent, so that \( \mathbf{P}(T_2 < \infty) = 1 \), and therefore \( \lim_{M \to \infty} \mathbf{P}(T_2 > M) = 0 \). Hence,

\[
\lim_{M \to \infty} \mathbf{E}[X_M | T_2 > M] = \lim_{M \to \infty} \frac{-[1 - \mathbf{P}(T_2 > M)] (2)}{\mathbf{P}(T_2 > M)} = -\infty.
\]

Hence, the statement is true and proved.