UNIVERSITY OF TORONTO

Faculty of Arts and Science

APRIL/MAY EXAMINATIONS 2002

STA447H1 S [Also cross-listed as STA2006H.]

Duration – 3 hours

NO AIDS ALLOWED.

(Number of questions: 10. Number of pages: 2. Total number of points: 105.)

1. (10 points) Consider a (discrete-time) Markov chain $\{X_n\}$ on the state space $S = \{1, 2, 3, 4\}$, with transition probabilities given by

$$(p_{ij}) = \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

Suppose $\mathbf{P}(X_0 = 4) = 1$. Compute $\mathbf{P}(X_2 = 3)$. (Explain your reasoning.)

2. (10 points) Either find (with explanation) an example of a Markov chain with state space $S = \{1, 2, 3\}$, and $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$, or prove that no such chain exists.

3. (10 points) Either find (with explanation) an example of a Markov chain with state space $S = \{1, 2, 3\}$, and $0 < f_{ij} < 1$ for all $i, j \in S$, or prove that no such chain exists.

4. (10 points) Let $S = \{1, 2, 3, \ldots\}$, and let

$$\pi_i = \begin{cases} 2^{-i+1}, & i \text{ even} \\ 8 \cdot 3^{-i-2}, & i \text{ odd} \end{cases}$$

Find (with explanation) transition probabilities (p_{ij}) for an irreducible, aperiodic Markov chain on S, such that π is stationary for (p_{ij}). [Hint: Don't forget the Metropolis Algorithm.]

5. (10 points) Consider an $M(\lambda) / M(\mu) / 1$ single-server queue. Let $Q(t)$ be the number of people in the system (i.e., waiting in the queue or being served) at time t. For $h > 0$ and non-negative integers i and j, let $p_{ij}(h) = \mathbf{P}[Q(t + h) = j \,|\, Q(t) = i]$.

(a) Find (with explanation) a value $r \geq 0$ such that $0 < \lim_{h\searrow 0} p_{58}(h) / h^r < \infty$.

(b) Compute (with explanation) the limit $\lim_{h\searrow 0} p_{58}(h) / h^r$, for the value of r found in part (a).

6. (10 points) Let $\{X(t)\}_{t\geq 0}$ be a continuous-time Markov process on the state space $S = \{1, 2, 3\}$, having generator given by

$$G = \begin{pmatrix} -3 & 3 & 0 \\ 0 & -2 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$

Find (with explanation) a stationary distribution for the process.

7. (10 points) Let $\{X_n\}_{n=0}^{\infty}$ be a discrete-time irreducible Markov chain on the state space $S = \{1, 2, 3, 4\}$, with $X_0 = 4$, having stationary distribution $\pi = (1/7, 1/7, 2/7, 3/7)$. Let $T_1, T_2, \ldots$ be the return times to the state 4. Let $Z_1, Z_2, \ldots$ be i.i.d. $\sim$ **Uniform**$[-12, 2]$. Let $N(t) = \max\{m \geq 0 : T_m \leq t\}$, and let $H(t) = \sum_{k=1}^{N(t)} Z_k$. Compute (with explanation) $\lim_{t\to\infty} H(t) / t$.

8. (10 points) Consider simple symmetric random walk $\{X_n\}$ on the set of all integers $\mathbf{Z}$, with $X_0 = 0$. For $m \in \mathbf{Z}$, let $T_m = \min\{n \geq 1;\ X_n = m\}$, and let $U = \min(T_4, T_{-6})$. Prove or disprove each of the following statements:

(a) $\mathbf{E}[X_U] = 0$.

(b) $\mathbf{E}[X_{T_4}] = 0$.

9. (10 points) Consider simple symmetric random walk on the set of all integers $\mathbf{Z}$, with $X_0 = 5$. Let $T = \min\{n \geq 0;\ X_{n+1} = X_n + 1\}$, and let $U = T + 1$. Prove or disprove each of the following statements:

(a) $\mathbf{E}[X_T] = 5$.

(b) $\mathbf{E}[X_U] = 5$.

10. (15 points) Let $S = \{0, 1, 2, \ldots, 9\}$. Define the Markov chain $\{X_n\}$ by: $X_0 = 7$; $p_{00} = p_{99} = 1$; $p_{89} = 1 - p_{87} = C_1$; and for $1 \leq i \leq 7$, $p_{i,i-1} = p_{i,i+2} = 1/2$. Let $T = \min\{n \geq 1 :\ X_n = 0 \text{ or } X_n = 9\}$. Let $Y_n = X_n - \min(n, T)\, C_2$.

(a) Find values of C_1 and C_2 such that $\{Y_n\}$ is a martingale.

(b) Use this to compute (with explanation) $E[Y_T]$.

(c) Use this to compute $E[T]$ in terms of $E[X_T]$.

END OF EXAM. TOTAL MARKS = 105.