STA 447/2006S, Spring 2002: Homework #1

Due by Monday, February 11, 4:00 p.m., in Sid Smith 6024.

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Copying other solutions is strictly prohibited!

[All questions are from the book Probability and Random Processes, Second Edition, by G.R. Grimmett and D.R. Stirzaker.]

INCLUDE YOUR NAME, STUDENT #, PROGRAM, AND YEAR.

REMINDER: Test #1 will be held in class on Thursday, February 14, beginning at 7:30 p.m. No aids will be allowed. You should bring your student card to the test.

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1. [10 points] Let X_n be the maximum reading obtained in the first n throws of a fair six-sided die. Show that X is a Markov chain, and find the n-step transition probabilities $p_{ij}^{(n)}$.

2. [15 points] Let X be a Markov chain with state space S, let $h : S \to T$ be a function, and let $Y_n = h(X_n)$.

(a) Suppose that h is a one-one function. Prove that $\{Y_n\}$ is a Markov chain on T.

(b) Suppose h is not one-one. Must $\{Y_n\}$ be a Markov chain? [NOTE: You must <u>prove</u> your assertion.]

3. [10 points] A particle performs a random walk on the (eight) vertices of a cube. At each step if remains where it is with probability $\frac{1}{4}$, or moves to each of its three neighbouring vertices with probability $\frac{1}{4}$ each. If the walk starts at a vertex v, find the mean number of steps until its first return to v.

4. [10 points] Show by example that chains which are not irreducible may have many different stationary distributions.

5. [10 points] **Superposition.** Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities λ and μ. Show that the arrivals of all flying objects form a Poisson process with intensity $\lambda + \mu$.

6. [20 points] Classify (as positive recurrent, null recurrent, or transient) the states of the discrete-time Markov chains with state space $S = \{1, 2, 3, 4\}$ and transition matrices

$$\text{(a)} \quad \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

7. [20 points] A transition matrix is called *doubly stochastic* if all its column sums equal 1, i.e. if $\sum_i p_{ij} = 1$ for all $j \in S$.

(a) Show that if a finite chain has a doubly stochastic transition matrix, then all its states are non-null persistent, and that if it is, in addition, irreducible and aperiodic then $p_{ij}(n) \to N^{-1}$ as $n \to \infty$, where N is the number of states.

(b) Show that, if an infinite irreducible chain has a doubly stochastic transition matrix, then its states are either all null persistent or all transient.

8. [15 points for parts (b) and (c) assuming the result of part (a), plus 10 bonus points if you also get part (a) which is more difficult]

(a) Show that for each pair i, j of states of an irreducible aperiodic chain, there exists $N = N(i, j)$ such that $p_{ij}(n) > 0$ for all $n \geq N$.

(b) Let X and Y be independent irreducible aperiodic Markov chains with the same state space S and same transition matrix P. Show that the bivariate chain $Z_n = (X_n, Y_n)$, $n \geq 0$, is irreducible and aperiodic.

(c) Show that the bivariate chain $\{Z_n\}$ may be reducible if X and Y are periodic.

9. [15 points] Consider the symmetric random walk in three dimensions on the set of points $\{(x, y, z) : x, y, z = 0, \pm 1, \pm 2, \ldots\}$; this process is a sequence $\{\mathbf{X}_n : n \geq 0\}$ of points such that $\mathbf{P}(\mathbf{X}_{n+1} = \mathbf{X}_n + \epsilon) = 1/6$ for $\epsilon = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$. Suppose that $\mathbf{X}_0 = (0, 0, 0)$. Show that

$$\mathbf{P}\Big(\mathbf{X}_{2n} = (0,0,0)\Big) = (1/6)^{2n} \sum_{i+j+k=n} \frac{(2n)!}{(i!j!k!)^2}$$

$$= (1/2)^{2n} \binom{2n}{n} \sum_{i+j+k=n} \left(\frac{n!}{3^n \, i!j!k!} \right)^2$$

and deduce by Stirling's formula that the origin is a transient state.