STA 447/2006S, Spring 2002: Homework #2

Due by Monday, March 18, 4:00 p.m., in Sid Smith 6024.

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Copying other solutions is strictly prohibited!

All questions except the last one are from the book Probability and Random Processes, Second Edition, by G.R. Grimmett and D.R. Stirzaker.

[INCLUDE YOUR NAME, STUDENT #, PROGRAM, AND YEAR.]

1. [10 points] **Thinning.** Insects land in the soup in the manner of a Poisson process with intensity λ, and each such insect is green with probability p, independently of the colours of all other insects. Show that the arrivals of green insects form a Poisson process with intensity λp.

2. [20 points] Let X be a Markov chain on $\{1, 2\}$ with generator

$$G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$$

where $\lambda\mu > 0$.

(a) Write down the forward equations and solve them [or, use eigenvalues and eigenvectors] to find the transition probabilities $p_{ij}(t)$ for $i, j \in \{1, 2\}$ and $t \geq 0$.

(c) Solve the equation $\pi G = 0$ in order to find the stationary distribution π. Verify that $p_{ij}(t) \to \pi_j$ as $t \to \infty$.

3. [10 points] As a continuation of the previous exercise, find

(a) $\mathbf{P}(X(t) = 2 \,|\, X(0) = 1,\, X(3t) = 1)$.

(b) $\mathbf{P}(X(t) = 2 \,|\, X(0) = 1,\, X(3t) = 1,\, X(4t) = 1)$.

4. [10 points] Planes land at Heathrow airport at the times of a renewal process with interarrival time having cumulative distribution function F. Each plane independently contains a random number of people which are i.i.d. with finite mean. Find an expression for the rate of arrival of passengers over a long time period, in terms of the mean number of passengers per plane, and the mean interarrival time μ.

5. [15 points] Let N be a Poisson process with intensity λ. Show that the total lifetime $D(t) = T_{N(t)+1} - T_{N(t)}$ at time t (i.e., the length of the interarrival time which contains t) has distribution function $\mathbf{P}(D(t) \leq x) = 1 - (1 + \lambda \min(t, x))e^{-\lambda x}$ for $x \geq 0$. Deduce that $\mathbf{E}(D(t)) = (2 - e^{-\lambda t}) / \lambda$. This is the 'inspection paradox'.

6. [10 points] A Type I counter records the arrivals of radioactive particles. Suppose that the arrival process is Poisson with intensity λ, and that the counter is locked for a dead period of fixed length T after each detected arrival. Show that the detection process $\widetilde{N}$ is a renewal process with interarrival time distribution $\widetilde{F}(x) = 1 - e^{-\lambda(x-T)}$ if $x \geq T$. Find an expression for $\mathbf{P}(\widetilde{N}(t) \geq k)$.

7. [20 points]

(i) A word processor has 100 different keys and a monkey is tapping them (uniformly) at random. Assuming no power failure, use the elementary renewal theorem to find the expected number of keys tapped until the first appearance of the sequence 'W. Shakespeare' (which contains 14 characters including the space).

(ii) Answer the same question for the sequence 'omo'.

8. [10 points] Let M and N be positive integers. Let $\{\widehat{X}_n\}$ be a discrete-time Markov chain on the state space $S = \{-M, -M+1, \ldots, -1, 0, 1, 2, \ldots, N\}$, such that $p_{0,1} = p_{0,-1} = 1/2$, and $p_{i,i+1} = 1$ for $1 \leq i \leq N-1$, and $p_{i,i-1} = 1$ for $-M+1 \leq i \leq -1$, and $p_{-M,0} = p_{N,0} = 1$. Let $\{X(t)\}_{t \geq 0}$ be the exponential holding time modification of $\{\widehat{X}_n\}$, with exponential holding time parameter λ. Compute $\lim_{t \to \infty} \mathbf{P}(X(t) = N)$. [Hint: Use W.L. Smith's Theorem.]