STA 447/2006S, Spring 2002: Homework #3

Due by Friday, April 12, 2:30 p.m., in Sid Smith 6024.

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Copying other solutions is strictly prohibited!

Reminder: Test #2 is on Thursday, March 28, in class. Bring your student card.

Reminder: Final Exam is on Monday, April 22, from 7:00 p.m. to 10:00 p.m., in Wetmore Hall of New College. Bring your student card.

INCLUDE YOUR NAME AND STUDENT NUMBER.

1. [10 points]: **Finite waiting room.** Consider an $M(\lambda)/M(\mu)/1$ queue with the constraint that arriving customers who see N customers in the line ahead of them leave and never return. Find the stationary distribution of the queue length.

2. [20 points]: **Baulking.** Consider an $M(\lambda)/M(\mu)/1$ queue with the constraint that if an arriving customer sees n customers in the line ahead of him, he joins the queue with probability $p(n)$ and otherwise leaves in disgust.

(a) Find the stationary distribution of the queue length if $p(n) = (n+1)^{-1}$.

(b) Find the stationary distribution π of the queue length if $p(n) = 2^{-n}$, and show that the probability that an arriving customer joins the queue (in equilibrium) is $\mu(1-\pi_0)\,/\,\lambda$.

3. [10 points]: Let X be a discrete-time Markov chain with countable state space S and transition matrix P. Suppose that $\psi : S \to \mathbf{R}$ is bounded and satisfies $\sum_{j \in S} p_{ij}\psi(j) \le \lambda\psi(i)$ for some $\lambda > 0$ and all $i \in S$. Show that $\lambda^{-n}\psi(X_n)$ constitutes a supermartingale.

4. [20 points]: Let $\{S_n\}_{n\ge 0}$ be simple symmetric random walk with $S_0 = 0$.

(a) Show that

$$Y_n = \frac{\cos\left\{\lambda[S_n - \frac{1}{2}(b-a)]\right\}}{(\cos\lambda)^n}$$

constitutes a martingale if $\cos\lambda \neq 0$.

(b) Let a and b be positive integers. Show that the time T until absorption at one of $-a$ and b satisfies

$$\mathbf{E}[(\cos\lambda)^{-T}] = \frac{\cos\{\frac{1}{2}\lambda(b-a)\}}{\cos\{\frac{1}{2}\lambda(b+a)\}}, \qquad 0 < \lambda < \frac{\pi}{b+a}.$$

5. [15 points]: Let $\{S_n\}_{n\geq 0}$ be simple symmetric random walk with $S_0 = 0$. For each of the following three random variables, determine whether or not it is a stopping time, and find its mean:

(a) $U = \min\{n \geq 5\,;\ S_n = S_{n-5} + 5\}$.

(b) $V = U - 5$.

(c) $W = \min\{n \geq 1\,;\ S_n = 1\}$.

THE REMAINING QUESTIONS ARE NOT TO BE HANDED IN.

6. [NOT TO BE HANDED IN]: Let $\{Y_n\}$ be a martingale with $\mathbf{E}[Y_n] = 0$ and $\mathbf{E}[Y_n^2] < \infty$ for all n. Show that

$$\mathbf{P}\left(\left(\max_{1\leq k\leq n} Y_k\right) > x\right) \leq \frac{\mathbf{E}[Y_n^2]}{\mathbf{E}[Y_n^2] + x^2}, \qquad x > 0.$$

7. [NOT TO BE HANDED IN]: Let $X_1, X_2, \ldots$ be independent random variables with

$$X_n = \begin{cases} 1 & \text{with probability } (2n)^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \\ -1 & \text{with probability } (2n)^{-1}. \end{cases}$$

Let $Y_1 = X_1$, and for $n \geq 2$,

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1} = 0 \\ nY_{n-1}|X_n| & \text{if } Y_{n-1} \neq 0. \end{cases}$$

Show that $\{Y_n\}$ is a martingale. Show that $\{Y_n\}$ does not converge almost surely. Does $\{Y_n\}$ converge in any way? Why does the martingale convergence theorem not apply?

8. [NOT TO BE HANDED IN]: **Pólya's urn.** A bag contains red and blue balls, with initially r red and b blue where $rb > 0$. A ball is drawn from the bag, its colour noted, and then it is returned to the bag together with a new ball of the same colour. Let R_n be the number of red balls after n such operations.

(a) Show that $Y_n = R_n/(n+r+b)$ is a martingale which converges a.s. and in mean.

(b) Let T be the number of balls drawn until the first blue ball appears, and suppose that $r = b = 1$. Show that $\mathbf{E}[(T+2)^{-1}] = 1/4$.

(c) Suppose $r = b = 1$, and show that $\mathbf{P}(Y_n \geq 3/4 \text{ for some } n) \leq 2/3$.

9. [NOT TO BE HANDED IN]: Let W be a standard Weiner process, that is, a process with independent increments and continuous sample paths such that $W(s+t) - W(s)$ is $N(0,t)$ for $t > 0$. Let α be a positive constant. Show that

(a) $\alpha W(t/\alpha^2)$ is a standard Weiner process.

(b) $W(t+\alpha) - W(\alpha)$ is a standard Weiner process.

10. [NOT TO BE HANDED IN]: Fix $\beta > 0$. Show that $U(t) = e^{-\beta t} W(e^{2\beta t} - 1)$ is an Ornstein-Uhlenbeck process if W is a standard Weiner process.