# STA 447/2006S, Spring 2002: Test #1: SOLUTIONS

**1.** For each of the following sets of conditions, either give (with explanation) an example of a valid transition matrix $(p_{ij})$ for a Markov chain on a state space $S$ which satisfies the conditions, or prove that no such Markov chain exists.

**(a)** (10 points) $3/4 < p_{12}^{(n)} < 1$ for all $n \geq 1$.

**Solution.** *There are many examples. If, say,* $S = \{1, 2\}$ *and*

$$(p_{ij}) = \begin{pmatrix} 1/8 & 7/8 \\ 0 & 1 \end{pmatrix},$$

*then* $p_{12}^{(n)} = 1 - (1/8)^n$, *so* $3/4 < p_{12}^{(n)} < 1$ *for all* $n \geq 1$.

**(b)** (10 points) $p_{11} > 1/2$, and the state 1 is transient.

**Solution.** *There are many examples. If, say,* $S = \{1, 2\}$ *and*

$$(p_{ij}) = \begin{pmatrix} 2/3 & 1/3 \\ 0 & 1 \end{pmatrix},$$

*then* $p_{11} = 2/3 > 1/2$. *However, once the chain hits* 2 *it will never return to* 1, *so that* $f_{11} \leq 1 - p_{12} = 2/3 < 1$. *Hence, state 1 is transient.*

**(c)** (10 points) $p_{11} > 1/2$, and the period of state 2 equals 2, and the chain is irreducible.

**Solution.** *This is impossible. Since* $p_{11} > 0$, *the period of state 1 equals 1. But if the chain is irreducible, then by a theorem from class, all states have the same period. So state 2 cannot have period equal to 2.*

**(d)** (10 points) $p_{12} = 0$ and $p_{12}^{(3)} = 0$, but $0 < p_{12}^{(2)} < 1$.

**Solution.** *There are many examples, such as*

$$(p_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad \text{or} \quad (p_{ij}) = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*In either case, it is easily computed (you must explain how!) that* $p_{12} = 0$, $p_{12}^{(2)} = 1/2$, *and* $p_{12}^{(3)} = 0$.

**(e)** (10 points) $f_{12} = 1/3$, and $f_{13} = 2/3$.

**Solution.** *This is easy, we can just set* $S = \{1, 2, 3\}$ *and*

$$(p_{ij}) = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Then after the first step, the chain just stays where it is. Hence, $f_{12} = p_{12} = 1/3$, and $f_{13} = p_{13} = 2/3$.*

**(f)** (10 points) $f_{12} = 1/2$, and $f_{13} = 2/3$.

**Solution.** *This is harder, but still possible. For example, suppose $S = \{1, 2, 3, 4\}$ and*

$$(p_{ij}) = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*Then the chain can only get to state 2 on the first move, so $f_{12} = p_{12} = 1/2$. The chain can get to state 3 only on either the first move (with probability $p_{13} = 1/2$) or the second move (with probability $p_{12}p_{23} = (1/2)(1/3) = 1/6$), so $f_{12} = f_{12}^{(1)} + f_{12}^{(2)} = 1/2 + 1/6 = 2/3$.*

**(g)** (10 points) $\lim_{n\to\infty} p_{21}^{(n)} = 1/3$.

**Solution.** *This is easy. For example, if $S = \{1, 2\}$ and*

$$P = (p_{ij}) = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix},$$

*then it is computed that $P^2 = P$. Hence, by induction, $P^n = P$ for all $n \geq 1$. This means that $p_{21}^{(n)} = p_{12} = 1/3$ for all $n \geq 1$, so of course $\lim_{n\to\infty} p_{21}^{(n)} = 1/3$.*

*Alternatively, we can set $S = \{1, 2, 3\}$, and let $P$ be any irreducible, aperiodic, doubly stochastic matrix. (You must give a specific example, and explain why it is irreducible and aperiodic!) Then from the first homework (or, from the Markov chain limit theorem and direct computation), we must have $\pi = (1/3, 1/3, 1/3)$ as a stationary distribution and $\lim_{n\to\infty} p_{21}^{(n)} = \pi_1 = 1/3$.*

**(h)** (10 points) $p_{12}^{(n)} \geq 1/4$ and $p_{21}^{(n)} \geq 1/4$ for all $n \geq 1$, and the state 1 is transient.

**Solution.** *This is impossible. If $p_{12}^{(n)} \geq 1/4$ and $p_{21}^{(n)} \geq 1/4$ for all $n \geq 1$, then for $k \geq 2$, we have by the Chapman-Kolmogorov equations that $p_{11}^{(k)} \geq p_{12}^{(1)} p_{21}^{(k-1)} \geq (1/4)(1/4) = 1/16$. Hence,*

$$\sum_{n=1}^{\infty} p_{11}^{(n)} \geq \sum_{k=2}^{\infty} p_{11}^{(k)} \geq \sum_{k=2}^{\infty} 16 = \infty.$$

*Hence, from a theorem in class, the state 1 must be recurrent, not transient.*