STA 447/2006S, Spring 2002: Test #2 SOLUTIONS

1. (10 points) Let $s(p, c, a)$ be the gambler's ruin probability, i.e. the probability that simple random walk with parameter p, started at $X_0 = a$, will hit c before it hits 0. Compute (with explanation) the limit $\lim_{n\to\infty} s(p, 2n, n)$, for $0 < p < 1$.

> **Solution.** *We recall from class that* $s(1/2, c, a) = a/c$, *and* $s(p, c, a) = \frac{((1-p)/p)^a - 1}{((1-p)/p)^c - 1}$ *for* $p \neq 1/2$. *Hence,* $\lim_{n\to\infty} s(1/2, 2n, n) = \lim_{n\to\infty} n/2n = \lim_{n\to\infty} 1/2 = 1/2$. *For* $p \neq 1/2$, $s(p, 2n, n) = \frac{((1-p)/p)^n - 1}{((1-p)/p)^{2n} - 1} = 1/[((1-p)/p)^n + 1]$. *If* $p > 1/2$ *then* $\lim_{n\to\infty}((1-p)/p)^n = 0$, *so* $\lim_{n\to\infty} s(p, 2n, n) = 1/(0+1) = 1$. *If* $p < 1/2$ *then* $\lim_{n\to\infty}((1-p)/p)^n = \infty$, *so* $\lim_{n\to\infty} s(p, 2n, n) = 1/(\infty + 1) = 0$.

2. (10 points) Let $\{\hat{X}_n\}_{n=0}^{\infty}$ be a discrete-time Markov chain on the state space $S = \{1, 2, 3\}$, with $\hat{X}_0 = 1$, and with transition probabilities given by

$$(\, p_{ij} \,) \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/6 & 1/2 \end{pmatrix}.$$

Let $\{X(t)\}_{t\geq 0}$ be the Exponential(λ) holding time modification of $\{\hat{X}_n\}$. Let $T_3 = \min\{t \geq 0 : \; X(t) = 3\}$. Compute (with explanation) the expected value of T_3.

> **Solution.** *From the matrix* $(\, p_{ij} \,)$, *we see that with probability 1 we will have* $\hat{X}_1 = 2$ *and* $\hat{X}_2 = 3$. *Hence,* T_3 *will be equal to the sum of two holding times of* $\{X(t)\}$. *But the holding times of* $\{X(t)\}$ *have distribution Exponential*(λ), *and hence mean* $1/\lambda$. *So,* $E[T_3] = \frac{1}{\lambda} + \frac{1}{\lambda} = 2/\lambda$.

3. (10 points) Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with parameter $\lambda = 3$. Let $X = N(8) - N(5)$, and let $Y = N(7) - N(2)$. Compute (with explanation) the value of $E[XY]$.

> **Solution.** *Let* $A = N(8) - N(7)$, $B = N(7) - N(5)$, *and* $C = N(5) - N(2)$. *Then* A, B, *and* C *are independent (since they are based on non-overlapping intervals). Also* $X = A + B$ *and* $Y = B + C$. *Hence,* $E[XY] = E[(A+B)(B+C)] = E[AB + AC + B^2 + BC] = E[A]E[B] + E[A]E[C] + E[B^2] + E[B]E[C]$. *Now,* $A \sim Poisson(3)$, $B \sim Poisson(6)$, *and* $C \sim Poisson(9)$. *Furthermore, we recall* that if* $Z \sim Poisson(\mu)$, *then* $E[Z] = \mu$ *and* $E[Z^2] = \mu + \mu^2$. *Hence,* $E[XY] = (3)(6) + (3)(9) + (6 + 36) + (6)(9) = 141$.

* *Or compute:* $E[Z] = \sum_{j=0}^{\infty} j e^{-\mu}\mu^j/j! = \sum_{j=1}^{\infty} e^{-\mu}\mu\,\mu^{j-1}/(j-1)! = \mu$; *and* $E[Z^2] = E[Z] + E[Z(Z-1)] = \mu + \sum_{j=0}^{\infty} j(j-1)e^{-\mu}\mu^j/j! = \mu + \sum_{j=2}^{\infty} e^{-\mu}\mu^2\mu^{j-2}/(j-2)! = \mu + \mu^2$.

4. (10 points) Let $\{X(t)\}_{t\geq 0}$ be a continuous-time Markov process on the state space $S = \{1, 2, 3, 4, 5\}$. Suppose it is known that for $0 \leq t \leq 0.03$,

$$P(X(t) = 2 \,|\, X(0) = 1) = 5t + 4t^2 + e^{3t} - 1\,.$$

Let $G = (\,g_{ij}\,)$ be the generator for this process. Compute g_{12}.

Solution. *By definition,*

$$g_{12} \;=\; \lim_{t\searrow 0} \frac{p_{12}(t)}{t} = \lim_{t\searrow 0} \frac{5t + 4t^2 + e^{3t} - 1}{t} \;=\; \frac{d}{dt}(5t + 4t^2 + e^{3t} - 1)\Big|_{t=0}$$

$$= \;(5 + 8t + 3e^{3t} - 0)\Big|_{t=0} \;=\; 5 + 0 + 3 - 0 \;=\; 8\,.$$

5. (10 points) Let $\{Y_n\}_{n=0}^{\infty}$ be i.i.d. $\sim$ $\mathbf{Uniform}[0, 10]$. Let $T_0 = 0$, and let $T_n = Y_1 + Y_2 + \ldots + Y_n$ for $n \geq 1$. Let $p = P[\,\exists\, n \geq 1 :\; 1234.5 < T_n < 1234.6]$. Find (with explanation) a good approximation to p.

Solution. *Let $Z_t = N(t + 0.1) - N(t)$ be the number of events between times t and $t + 0.1$. Then $p = P[Z_{1234.5} \geq 1]$. Now, Blackwell's Renewal Theorem says that $\lim_{t\to\infty} E[Z_t] = 0.1/\mu$, so $E[Z_{1234.5}] \approx 0.1/\mu$. Since $Y_n \sim \mathbf{Uniform}[0, 10]$, we have $\mu = E[Y_n] = 5$, so $E[Z_{1234.5}] \approx 0.1/5 = 0.02$. Finally, since $P[Z_{1234.5} \geq 2]$ is extremely small, we see that $Z_{1234.5}$ is approximately an indicator function, so that $p = P[Z_{1234.5} \geq 1] \approx E[Z_{1234.5}] \approx 0.02$.*