On Duality of Probabilities for Card-dealing

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Milton Sobel and Krzysztof Frankowski [1] recently described a notion of "duality" for card-dealing probabilities. They prove the equality of certain probabilities for different card-dealing schemes, and then discuss applications of their work. Their proof is quite computationally involved. The purpose of this short note is to present a much simpler and more conceptual proof of their result, which also generalizes it substantially.

Consider a deck of N cards, called Deck #1, which contains b different "categories" (e.g. suits), of sizes $m_1, \ldots, m_b$. Consider dealing (uniformly at random) j different hands from this deck, of sizes $n_1, \ldots, n_j$. Without loss of generality we assume that $m_1 + \ldots + m_b = n_1 + \ldots + n_j = N$ (since if not, we could always include an extra category or an extra hand to make this so). For clarity, the case $N = 52$, $b = j = 4$, $m_1 = \ldots = m_4 = n_1 = \ldots = n_4 = 13$, corresponds to dealing an ordinary bridge game.

For $1 \leq x \leq b$ and $1 \leq y \leq j$, we let C_{xy} be the random variable representing the number of cards from category x in hand y. Let C record these numbers as a $b \times j$ matrix.

To explain the duality notion, consider a second deck of N cards, Deck #2, which has the sizes of categories and hands reversed. Thus, this second deck has j different categories of sizes $n_1, \ldots, n_j$, and we deal out b different hands of sizes $m_1, \ldots, m_b$. We let D_{xy} be the random variable representing the number of cards from category x in hand y. Let D record these numbers as a $j \times b$ matrix.

Theorem. *The matrix D has the same probability distribution as does the transpose of the matrix C. In symbols,*

$$D \stackrel{\mathrm{d}}{=} C^t \quad .$$

Remark. In [1], only the case $m_1 = \ldots = m_{b-1}$ and $n_1 = \ldots = n_{j-1}$ is considered. It is proved there that

$$P\left(\max_{\substack{1\leq x\leq b-1\\1\leq y\leq j-1}} C_{xy} \leq r\right) = P\left(\max_{\substack{1\leq x\leq j-1\\1\leq y\leq b-1}} D_{xy} \leq r\right)$$

for any r (and similarly with $\leq$ replaced by $\geq$). Our theorem thus generalizes [1] in two ways. Firstly, we allow unequal category and hand sizes. Secondly, we prove the equality of the full joint distributions, not just the equality of specific probabilities.

Example. For example, in the case of an ordinary bridge game, the theorem says that the probability distribution of the four suits in any given hand is equal to the probability distribution of spades among the four hands. In particular, it shows [1] that the probability that a given hand has no more than 4 cards of any one suit is the same as the probability that none of the hands contain more than 4 spades.

Proof of the theorem. We describe here a simple proof of this theorem. Our proof is based on a coupled construction for dealing out the two decks simultaneously, under which the matrices D and C^t are actually *equal*. (This is a standard method of showing equality in distribution.)

Let Deck #1 and Deck #2 be as above. Deal the cards from Deck #1 out uniformly at random (in any order), producing j hands of sizes $n_1, \ldots, n_j$.

Deal the cards from Deck #2 simultaneously, as follows: *Each time a card of category x is dealt to hand y in Deck #1, deal a card from category y to hand x in Deck #2.*

When Deck #1 is completely dealt out, Deck #2 will also be completely dealt out, into b hands of sizes $m_1, \ldots, m_b$. Also, it is easily checked that Deck #2 will have been dealt out with the "proper probabilities", i.e. with the cards dealt out uniformly at random. Thus, we have constructed a coupling for dealing out the two decks.

However, it is easily seen that for $1 \le x \le j$ and $1 \le y \le b$,

$$\begin{aligned}&(\#\text{ cards of category } x \text{ in hand } y \text{ for Deck } \#1)\\ &\qquad\qquad = (\#\text{ cards of category } y \text{ in hand } x \text{ for Deck } \#2.)\end{aligned}$$

This shows that for this coupling, we have $D = C^t$. The result about distributions follows. ∎

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REFERENCE

[1] M. Sobel and K. Frankowski (1992), "A Duality Theorem for Solving Multiple-Player Multivariate Hypergeometric Problems". Technical Report, Department of Computer Science, University of Minnesota.