Mathematics to Scale

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(June, 2016)

Mathematics is a wonderful subject, but much of it requires years of study to fully appreciate. When I was recently asked to design a course in quantitative reasoning for Humanities students [8], I had to come up with mathematical ideas and examples which were interesting, relevant, exciting, and most importantly, comprehensible without much math background.

Now, I do have some experience communicating mathematical ideas to non-mathematicians, including writing a bestselling general-interest book about probabilities [1], helping to uncover a major front-page lottery scandal [6], and writing about such diverse topics as the mathematics of music [2], probability and justice [7], sports statistics [5], the Monty Hall problem [3], and even family relationships [4]. But beyond that, what *fundamental* mathematical idea could I use to begin the course, to show these Humanities students the interest and power of mathematical thinking, without scaring them off or pushing them towards feelings of math anxiety?

After much consideration, I decided to begin my course with the concept of **scaling**. That is, how do various quantities change when an object's size is modified?

The key to scaling is that it depends on the *dimension*. Consider first a **one-dimensional object** like a line. If you wanted to draw a second line which was twice as long as your first line, then how many times as much ink would it require? Why, twice as much, of course. Indeed, in one dimension, an object's length or size or amount of ink are all pretty much the same thing, and there is little more to say.

In **two dimensions**, the situation is more interesting. Suppose you draw a square on a page, and then later wish to draw a second square which is twice as big (in all directions). How many times as much ink do you require now? Well, *four* times as much. This is because in two dimensions, an object's area is proportional to its length times its height, and if an object is expanded then each of its length and height is multiplied by two, so the product is multiplied by $2 \times 2 = 4$. Similarly, to draw a square three times as large (in all directions) would require $3 \times 3 = 9$ times as much ink.

The importance of this observation is that it actually **applies to more than just squares**. Indeed, *any* shape drawn on a page can be thought of as being made up of lots and lots of tiny disjoint squares. (Formally, this is justified by *calculus*, upon taking the *limit* of more and more smaller and smaller squares.) So, consider *any* shape drawn on a page (say, a child's drawing of a heart on a mother's day card). If you wanted to draw a new version which was identical except twice as large (in all directions), then it would require $2 \times 2 = 4$ times as much ink. This is true regardless of the chosen shape, and regardless of its actual area (which probably couldn't be computed precisely anyway).

More generally, if any two-dimensional region is stretched by one factor vertically and another factor horizontally, then the ratio of the areas is simply the product of the two stretch factors. For example, **dimes** are approximately one centimeter by one centimeter. So, about how many dimes could be taped flat against my classroom's wall, which is about four meters tall and twelve meters wide? Well, here the vertical scaling factor is 400, and the horizontal factor is 1,200, so the total number of dimes is $400 \times 1,200 = 480,000$, worth a total $48,000. That's a lot of dimes.

For another example, suppose **flowers** are planted about 20 centimeters apart. Then about how many flowers are on a five meter by eight meter field? Well, roughly speaking, here each flower occupies a disjoint "region" of about 20 centimeters by 20 centimeters. And the entire field can be viewed as a scaled up version of one such region. So, the number of flowers on the whole field is the product of the two scaling factors, namely $(500/20) \times (800/20) = 25 \times 40 = 1,000$. So, there are about a thousand of them – lots of flowers!

Next, consider scaling in **three dimensions**. If we have a three-dimensional cube, and then expand it to be twice as large (in all directions), then it has *three* different scaling factors, so its volume is multiplied by $2 \times 2 \times 2 = 8$. That is, it is eight times as large!

And once again, this principle **applies to more than just cubes**, since *any* volume can be thought of as consisting of lots and lots of tiny disjoint cubes. For example, consider a **glass of beer** (a very relevant example for students!). If a second glass is twice as large in all directions, then it holds *eight times* as much beer – a fact which surprises many people. Or, suppose instead that a second glass is twice as tall, but only 2/3 as wide and deep. Most people would think the second glass holds more beer. But in fact, it holds $2 \times (2/3) \times (2/3) = 8/9$ times as much. And $8/9 < 1$, so the second glass actually holds *less*.

Even more dramatically, consider a **cone-shaped glass** (like a fancy wine glass, or certain water-cooler cups). Suppose it is filled up to 2/3 of its full height. Then what fraction of its volume is full? Well, since it is cone-shaped, its bottom two-thirds is identical to the entire cup, except scaled by 2/3 in all directions. So, its bottom two-thirds holds $(2/3) \times (2/3) \times (2/3) = 8/27 \doteq 30\%$ of its full volume. So, with a cone-shaped cup, if the bartender fills 2/3 of its height, he is only giving you about 30% of a full glass of wine. Tell him to fill it up!

Another interesting application is to **mass**. A standard reference point is that a 10 cm $\times$ 10 cm $\times$ 10 cm cube of water equals one litre, and weighs one kilogram. So what about a 1 m $\times$ 1 m $\times$ 1 m cube of water? Well, 1 m is ten times as long as 10 cm. So, a 1 m $\times$ 1 m $\times$ 1 m cube has volume $10 \times 10 \times 10 = 1,000$ litres, and weighs 1,000 kilograms (about 2,205 pounds) – much too heavy to lift! Similarly, a 160 cm $\times$ 200 cm $\times$ 20 cm waterbed has volume $16 \times 20 \times 2 = 640$ litres, and weighs a massive 640 kilograms (about 1,411 pounds). This is why waterbeds can only be put into sturdy houses, and cannot be moved without first being drained.

Another perspective comes from picturing **crowds of people**. If you are in a one-dimensional *lineup* of people spaced about 0.5 meters apart, then the number of people standing within ten meters of you (in front or behind) is about $20/0.5 = 40$. But if you are in a two-dimensional *crowd* of people (at a concert or party or dance), all spaced about 0.5 meters apart, then the number of people within ten meters of you is about $(20/0.5) \times (20/0.5) = 1,600$ – lots more! (Or, if you really want to consider just a *circle* of people around you, not a square, then the answer is more like $\pi(10/0.5)^2 \doteq 1,257$.) Or, if a flock of *birds*, flying in *three* dimensions, are spaced about 0.5 meters apart, then the number of birds within ten meters of any one (central) bird is about $(20/0.5) \times (20/0.5) \times (20/0.5) = 64,000$, a massive number. That's scaling, in different dimensions.

Similarly, **stars** in our galaxy are approximately five light-years apart on average. So if stars are visible up to, say, one hundred light-years away in all directions, then the number of visible stars is roughly $(200/5) \times (200/5) \times (100/5) = 32,000$. (Or, if you want to count only a half-sphere of visibility, then it's about $\pi(100/5)^3/2 \doteq 12,566$.) Starry starry night, indeed! So, scaling explains why on a clear night you can see so many stars, even though most stars are too far away to be visible.

Comparing two different **people** is also fun. Suppose that Person #2 is twice (say) as large as Person #1 in all directions, otherwise identical. Then how many times as high can Person #2 reach? Answer: 2 (since height is one-dimensional). How many times as much does Person #2 weigh? Answer: 8 (since weight is three-dimensional). How many times as many blades of grass does Person #2 trample if they each take one step on a lawn? Answer: 4 (since foot area is two-dimensional). How many times as much blood does Person #2's arteries contain? Answer: 8 (since volume is three-dimensional). How many times as much skin surface (e.g. for tattoos) does Person #2 have? Answer: 4 (since skin surface is two-dimensional). And so on. Again, these answers do not depend on the precise shape of the people or their feet or arteries, they just involve scaling. (And of course, similar answers apply to other scaling factors besides two.)

Next, consider **pressure**, i.e. the amount of force per unit area on e.g. the ground underneath our feet. Considering **snowshoes** is instructive. Snowshoes work by spreading our mass over a larger area, to reduce the pressure on each individual spot of snow. Suppose I am wearing snowshoes which are twice as wide, and three times as long, as my regular boot. Then my same mass is spread over an area which is $2 \times 3 = 6$ times as large. So, the amount of pressure I exert on any one spot of snow is only $1/6$ as large. This is why I can (hopefully) stand on the top of the snow while wearing snowshoes, even if I would have sunk down deep in my normal boots.

Consider now a **giant lizard** (like Godzilla?). Suppose the giant is one thousand times as large as a regular lizard, in all directions. Then it weighs $1,000 \times 1,000 \times 1,000 = 1,000,000,000$ (one billion) times as much (!). And it has one billion times as much blood, and so on. On the other hand, its foot area is $1,000 \times 1,000 = 1,000,000$ (one million) times as large. So, the amount of *pressure* that it exerts on the ground is $1,000,000,000$

divided by $1,000,000$, or $1,000$ (one thousand) times as much. That is why giants tend to crush whatever they step on. But because of scaling, the pressure is only multiplied by their scaling factor (e.g. a thousand), not by their full weight ratio (e.g. a billion).

Finally, consider **rainfall**. You might have noticed that rain amounts are normally reported as *lengths*, e.g. "downtown Toronto received 40.6 mm of rain today". How could this be? Well, consider two different bins left out during a rainstorm. Suppose the second bin is twice as large as the first (in all directions). Then the *area* at the top of the bin is $2 \times 2 = 4$ times as large. So, the volume of rain collected by the second bin is 4 times as much. On the other hand, this rain is then spread over a bass area which is also $2 \times 2 = 4$ times as large. This means that the *height* of the rain in the second bucket is $4/4 = 1$ times as high, i.e. the height of rain in the two buckets is identical! Similar considerations apply to any other buckets of any other sizes and shapes, as long as they have straight sides (so the area of each bucket is the same at the top and the bottom). So, when reporting an amount of rainfall as a length, that length is equal to the height of water that would be left in *any* straight-sided bucket of any size whatsoever. And *that* is why rainfall amounts can be reported as simple lengths, not as "volume per unit bucket" or some other complicated standard.

So, the next time someone asks you for an easy-to-understand example of how mathematical thinking applies to everyday life, tell them a tale or two about scaling!

References

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