Waiting Time Correlations on Disorderly Streetcar Routes

by

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(February, 2007)

Abstract. We propose a model for disorderly streetcar routes. Through simulations and comparisons to real data, we illustrate that such routes may sometimes become supersaturated, in the sense that forward and backward waiting times become positively correlated.

1. Introduction.

It has long been known [7] that public transportation routes can produce unstable passenger servicing. This short paper considers the following question. Suppose a commuter attempts to catch a streetcar, but arrives at the stop just slightly too late. How does this near miss affect the waiting time until the next streetcar arrives?

If the streetcars arrive on a regular schedule, then clearly just missing one streetcar increases the waiting time until the next. For example, if streetcars arrive once every ten minutes, and the commuter arrives at a random time, then their mean wait is five minutes, but if they arrive one minute after a streetcar passes then their wait is nine minutes. The time since the previous streetcar is perfectly negatively correlated with the time until the next one.

At the other extreme, if the streetcars are completely independent and random, then their arrival times will converge to a Poisson process (see e.g. [1], [4]). In the limit, the arrival times are then Markovian, i.e. the distribution of the future is completely independent of the past. The time since the previous streetcar has zero correlation with the time until the next one.

This paper considers a third case. The streetcars are assumed to move at random velocities, and thus to become more and more disorderly over time. However, as on real streetcar

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routes, the streetcars are unable to pass each other, and must slow down if they get too close to the streetcar in front of them. Under such a model, how is the time until the next streetcar affected by the time since the last one?

We present a model for disorderly streetcar routes in Section 2. Section 3 contains a simulation model, while Section 4 describes real data from a real streetcar route near the University of Toronto. Our results are presented in Section 5, and a brief conclusion is in Section 6.

2. The Model.

We first define a *non-interfering* model, as follows. We assume that streetcar i has position $x_i(t) \in \mathbf{R}$ at time t, (ordered initially so that $x_i(0) \le x_{i+1}(0)$ for all i), and velocities $v_i(t) \in [0,1]$. At time $t = 0$, the streetcars are equally spaced (with inter-car distance $I \le 1$), each starting with its maximal velocity $v_i(0) = 1$. Thus,

$$x_i(t) \;=\; x_i(0) + \int_0^t v_i(s)\, ds\,.$$

It remains to specify $v_i(t)$. The *orderly* version has $v_i(t) = 1$ for all i and t, so the streetcars remain equally spaced forever, and the backward and forward waiting times are perfectly negatively correlated. In the *disorderly* version, the $v_i(t)$ instead form independent scaled Brownian motions constrained to $[0,1]$. That is, $v_i(0) = 1$, and the $\{v_i(t)\}_{t\ge 0}$ are independent Brownian motions each with volatility σ, reflected at 0 and at 1 to remain in the interval $[0,1]$.

The *interfering model* is similar to the above, except we assume in addition that no streetcar can get within Δ of another (where $\Delta < I$). That is, we impose the additional *interfering restriction* that:

$$x_i(t) \;\le\; x_{i+1}(t) - \Delta\,. \tag{1}$$

Specifically, $v_i(t)$ is reduced to $v_{i+1}(t)$ whenever $x_i(t) = x_{i+1}(t) - \Delta$. (Intuitively, the non-interfering model corresponds to buses which can pass each other, while the interfering model corresponds to streetcars that cannot. In the orderly model, the interfering condition is irrelevant.)

In either model, for those i with $x_i(0) < 0$, let $a_i = \inf\{t > 0 : x_i(t) \ge 0\}$ be the arrival time of streetcar i at position 0. For $t > 0$, let $F_t = \min_i\{a_i - t \,|\, a_i \ge t\}$ be the time forward from t until the next arrival, and let $B_t = \min_i\{t - a_i \,|\, a_i \le t\}$ be the time from t back to the previous streetcar. Let U be uniformly distributed over the interval $[t_0, t_0+1]$, representing a commuter's random arrival time within a specified one-hour interval (after t_0 hours of "burn in"), and let ϵ be a small positive quantity.

We are interested in the following quantities: $\mathbf{E}(F_U)$, the commuter's mean waiting time; $\mathrm{Corr}(F_U,\, B_U)$, the correlation of the commuter's forward and backward waiting times; and $\mathrm{PC} = 100 \times \mathbf{P}[F_U < \epsilon \,|\, B_U < \epsilon] \;/\; \mathbf{P}[F_U < \epsilon]$, the percentage change in the commuter's probability of seeing another streetcar very soon given that they just missed the previous one.

For the orderly model ($v_i(t) \equiv 1$), the streetcars will be equally spaced, the interfering condition is irrelevant, and we will have $\mathrm{Corr}(F_U,\, B_U) = -1$ and $\mathbf{P}[B_U < \epsilon \,|\, F_U < \epsilon] = 0$ for any $\epsilon < I/2$. That is, the longer since the previous streetcar, the shorter until the next one, corresponding to our usual intuition.

For the disorderly model *without* the interfering condition (1), i.e. where the $\{v_i(t)\}$ are independent Brownian motions without restriction, the streetcar positions on bounded intervals will converge (as $t_0 \to \infty$) to a Poisson process. So, for large t_0, we will have $\mathrm{Corr}(F_U,\, B_U) \approx 0$, and $\mathbf{P}[B_U < \epsilon \,|\, F_U < \epsilon] \approx \mathbf{P}[B_U < \epsilon]$. That is, in the limit, the distance to the previous streetcar will have no effect whatsoever on the distance to the next one in this case.

But what about the interfering disorderly model? In this case, the streetcars will tend to bunch up together. But will this bunching be more or less than the Poisson limit for the non-interfering case? In particular, can $\mathrm{Corr}(F_U,\, B_U)$ actually be *positive*, so the longer since the last streetcar means the *longer* until the next one. This is what we investigate below. Most streetcar routes will be somewhere between the orderly and the Poisson states; if a route exceeds this restriction (for example, by having $\mathrm{Corr}(F_U,\, B_U) > 0$), then we will call it *supersaturated*.

Remark. Our model is somewhat similar to the simple exclusion process, which is known to exhibit clustering effects (see e.g. [5]). However, our model depends on additional parameters like I and Δ and σ, and it is not clear that theoretical studies of the simple exclusion process provide insight into such quantities as $\mathrm{Corr}(F_U,\, B_U)$. We thus proceed to simulation studies.

3. Simulations.

To do simulations on the interfering Brownian streetcar model as above, we consider the following corresponding discrete model. Initially, the $x_i(0)$ are equally spaced with inter-car distance I, and $v_i(0) = 1$ for all i. Then, for small $\delta > 0$, and for $t = 0, \delta, 2\delta, 3\delta, \ldots$,

$$v_i(t+\delta) \;=\; R(v_i(t) + \delta^{1/2}\, Z_{it})$$

where $\{Z_{it}\}$ are i.i.d. Normal$(0, \sigma^2)$, and where $R(\cdots)$ is the reflection operator which reflects any real number into the interval $[0, 1]$, i.e. $R(x)$ is defined recursively by

$$R(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ R(2 - x), & x > 1 \\ R(-x), & x < 0 \end{cases}$$

Then, iteratively over i,

$$x_i(t + \delta) = x_i(t) + \delta\, v_i(t)\,. \tag{2}$$

Then, in accordance with (1), whenever (2) makes $x_i(t+\delta) > x_{i+1}(t+\delta) - \Delta$, we immediately "slow down" car i by setting $x_i(t + \delta) = x_{i+1}(t + \delta) - \Delta$ and $v_i(t + \delta) = v_i(t + \delta)$. Finally, to accommodate the finiteness of computers, we simulate the streetcars only over a finite interval $[-M, 0]$ (where $M > 1$), with periodic boundary (so we subtract M from streetcar positions whenever they go positive). This corresponds to a cyclic route of length M, which is a reasonable model for real streetcar routes, and in any case should not significantly affect the joint distribution of F_t and B_t.

All of the arrival times between t_0 and $t_0 + 1$ are recorded, along with the last arrival time before t_0 and the first arrival time after $t_0 + 1$. This allows us to compute F_U and B_U for a particular value of U. Simulating many different values of U then allows us to estimate $\text{Corr}(F_U, B_U)$ as well as related quantities such as $\mathbf{E}(F_U)$ and $\mathbf{P}(F_U < \epsilon \,|\, B_U < \epsilon)$. [The software and data are available at probability.ca/streetcar.]

Remark. Using $x_{i+1}(t + \delta)$ instead of $x_{i+1}(t)$ above corresponds to doing the iteration in reverse order, i.e. updating x_i for larger i first. This seems appropriate to allow different streetcars to move together without being artificially pulled apart. Of course, this issue becomes less and less relevant as $\delta \to 0$.

In our simulations, we set $I = 1/28$, corresponding to 28 streetcars per hour on average; see Section 4. We also set $\Delta = I/20$ (i.e., no streetcar may come within 1/20 of the mean inter-car distance of another), $\epsilon = 1/120$ (corresponding a streetcar arriving within 30-seconds), and $M = 3$.

For the velocity volatility parameter σ, we consider both 0.01 and 0.005, as well as the orderly model with $\sigma = 0$. For the number of hours t_0 over which the system becomes disorderly, we considered 3, 8, and 24. We considered both the interfering model and the non-interfering model.

For each simulation, standard errors were computed by running the identical simulation multiple independent times. The results are presented in Section 5.

4. Real Data.

As a check of our model, we observed actual arrival times for the Spadina Avenue streetcar, adjacent to the University of Toronto campus, during the afternoon rush hour. This streetcar route has frequent but erratic service, making it an appropriate case study for our disorderly streetcar model.

We recorded the (northbound) arrival times between 4:00 and 5:00 p.m., as well as the last arrival before 4:00 and the first arrival after 5:00, on each of two different weekdays (called A and B). The number of arrivals between 4:00 and 5:00 on the two days were 28 and 29, respectively, motivating the choice of $I = 1/28$ in Section 3 above.

We then mapped the hour from 4:00 to 5:00 linearly onto $[0, 1]$. This allowed us to compute forward and backward waiting times, F_t and B_t, for either data set, for any $t \in [0, 1]$. We were thus able to compute $\text{Corr}(F_U, B_U)$ etc. for $U \sim \text{Uniform}[0, 1]$, and to compare the real data to simulations from our streetcar model.

5. Results.

The results of the simulations and real data are presented in the following table:

Row #	σ	t_0	Int.	$\mathbf{E}(F_U)$	$\text{Corr}(F_U, B_U)$	PC
1	0	any	any	0.017857	-1	0
2	0.01	3	Y	0.02718 ± 0.00050	-0.245 ± 0.011	64.4 ± 2.1
3	0.01	8	Y	0.0544 ± 0.0017	0.010 ± 0.023	172.8 ± 7.3
4	0.01	24	Y	0.1185 ± 0.0080	0.127 ± 0.054	262.0 ± 5.3
5	0.005	8	Y	0.0387 ± 0.0011	-0.104 ± 0.022	123.8 ± 5.1
6	0.01	8	N	0.03141 ± 0.00087	-0.162 ± 0.029	85.2 ± 6.6
7	0.01	24	N	0.0360 ± 0.0012	-0.059 ± 0.032	91.9 ± 2.6
8	Real Data A			0.02985 ± 0.00007	-0.0438 ± 0.0018	74.0 ± 1.3
9	Real Data B			0.02771 ± 0.00012	0.0109 ± 0.0031	84.50 ± 0.99
10	Poisson limit			0.03571	0	100

Table 1: Results of the simulations and data analysis. Here σ is the volatility of the velocities, t_0 is the burn-in time, and "Int." indicates whether or not the interfering model is assumed. All other parameters are as specified in Section 3. The final three columns are the results: $\mathbf{E}(F_U)$ is the mean wait time, $\text{Corr}(F_U, B_U)$ is the waiting time correlation, and PC is the percentage change in the probability that $F_U < \epsilon$ as above. Standard errors are also given where appropriate.

In Table 1, Row 1 corresponds to orderly velocities, with no randomness. In this case, the mean waiting time $\mathbf{E}(F_U)$ is simply $I/2$, the correlation of forward and backward waiting times is -1, and PC $= 0$ because it is impossible to have $F_U < \epsilon$ and $B_U < \epsilon$ simultaneously.

By contrast, Row 10 is the Poisson process limit corresponding to buses that arrive randomly completely independently of each other. In that case, as is well known, the expected waiting time is I, twice what it would be in the orderly case. Meanwhile, the forward-backward correlation is zero, and PC $= 100$, both reflecting the fact that the future is independent of the past.

Rows 6 and 7 are simulations without the interference condition, and thus represent gradually disordering buses. At earlier times (Row 6), the results are somewhere between the initial orderly state (Row 1) and the limiting Poisson state (Row 10), while at later times (Row 7) they are getting closer to the latter.

Rows 2, 3, and 4 present our main focus, disorderly interfering streetcar routes, at three different times. At earlier times (Row 2) the results are again somewhere between the orderly and Poisson states. But at later times (Rows 3 and 4), the results *surpass* the corresponding Poisson values in all respects: the expected wait times become even longer, the forward-backward correlations become positive, and PC significantly exceeds 100.

This illustrates that the disorderly streetcar route can lead to *supersaturated* states, wherein the clumping of streetcars is greater than for the independent Poisson limit. The result is even less efficient, in that expected wait times are even larger. More interestingly, smaller values of B_U now correspond to *smaller* values of F_U. If a commuter just misses one streetcar, then they will have a *smaller* average wait time until the next one, and a *larger* probability of another streetcar arriving very soon.

Row 5 illustrates the effect of reducing the randomness factor σ. Comparing Rows 5 and 3 indicates, as expected, that if σ is smaller, then it takes longer for the route to move away from its initial orderly state. As a result, the expected wait time remains smaller, and the forward-backward correlations remain more negative, for a longer period of time.

Finally, Rows 8 and 9 are for the real data described in Section 4. Data A (Row 8) is again somewhere between the orderly and Poisson states, with a forward-backward correlation that is just slightly negative. However, Data B (Row 9) has indeed surpassed the Poisson limit, at least in the sense of having positively correlated forward and backward waiting times. Even Data B is not truly supersaturated, since its expected waiting time and PC value are both still somewhere between orderly and Poisson. However, its positive correlation value does indicate that some aspects of supersaturation are indeed possible on real streetcar routes. Presumably, an even more disorderly and random streetcar route (if one exists) would exhibit this supersaturation to a greater extent.

Remark. This clumping of streetcars is somewhat analogous to the tendency of buses and elevators to pair up once they get behind schedule due to the increasing time spent picking up passengers; see e.g. [6], [3], [2].

6. Conclusion.

This paper has argued that because streetcars interfere with each other by disallowing passing, disorderly streetcar routes can become *supersaturated*, in the sense that their descriptive parameters (mean waiting time, forward-backward correlation, and percentage change of probability of another arrival within time ϵ) can exceed those of the independent Poisson limit that would apply without the interference condition.

We have supported this assertion with simulations of a mathematical model for disorderly streetcar routes. The simulations have confirmed that each of those three descriptive parameters can indeed exceed the Poisson limit.

We have also compared our simulation results to real data from an actual streetcar route near the University of Toronto. We found that while the real data did not completely support supersaturation, it did in one case show a positive correlation between forward and backward waiting times, which could not occur in the non-interfering model. Furthermore, while the other descriptive parameters were somewhat less than for the Poisson limit, they were still far closer to Poisson than to a corresponding orderly route.

Our conclusion can be stated informally as saying that on a very disorderly streetcar route, if a commuter just misses one streetcar, then this is not necessarily bad news. Indeed, counter to intuition, it may actually *reduce* their mean waiting time until the next streetcar arrives.

Acknowledgements. I thank Margaret Fulford for assistance with collecting the data.

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