Nash Equilibria for Voter Models with Randomly Perceived Positions

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(Version of July 13, 2017)

Abstract. We introduce a voter model in which parties' intended policy positions are perceived by voters with some random uncertainty. We prove that for a total of three parties, under some mild assumptions, this model has a Nash equilibrium in which all three parties attempt to contest the election with the median policy. This contrasts with Duverger's Law which asserts that only two parties will contest the election at all, consistent with some different voter models.

Keywords: Voter model; Game theory; Nash equilibrium; Multiple parties; Uncertain position.

Mathematics Subject Classification (2010): Primary 91A06; Secondary 60E15, 91F10.

1 Introduction

In game theory, each "player" chooses certain actions, and receives a resulting payoff (see e.g. Osborne, 2003). The collection of actions is a *Nash equilibrium* (Nash, 1951) if no one player can change their action in a way which increases their expected payoff if the other players' actions remain unchanged.

Among many other applications, Nash equilibria are used to model strategies of political parties in elections. In one standard model, attributed variously to Hotelling (1929) and Downs (1957) and others, political positions are mapped to the real line, and voters' opinions follow some fixed probability density v on the real line, and each of n different political parties stakes out some position $x_i \in \mathbf{R}$ (or chooses not to contest the election at all, by setting $x_i = \text{OUT}$). Then, each voter is assumed to vote for whichever party's position is closest to their own. The payoff for each political party is 0 if they do not contest the election, or 1 if they receive the most votes, or $1/k$ if they tie for most votes with a total of k different parties, or -1 if they contest the election and receive fewer votes than some other party. In this way, each of the n political parties are a "player" in a game, with incentive to choose a political position only if they can attain (or tie for) the most votes.

With just $n = 2$ parties, the Nash equilibrium for this model is straightforward. Namely, each of the two parties will contest the election with political position equal to the median of v, i.e. $x_1 = x_2 = m$ where $\int_{-\infty}^{m} v(t)\,dt = 1/2$. In this way, they each receive payoff of $1/2$.

And, if either party deviates to another position $x_i \neq m$, then they will receive fewer votes than the other party, and thus receive payoff of $-1 < 1/2$. Or, if either party deviates to $x_i = \mathrm{OUT}$, then they will receive payoff of $0 < 1/2$. So, neither party has anything to gain by deviating from the position $x_i = m$, confirming that it is a Nash equilibrium.

However, if $n \geq 3$, then the situation becomes much more complicated. For example, if the first two parties each choose the median position $x_1 = x_2 = m$, then the third party can deviate to slightly more than m, thus winning nearly half of the votes while the other two parties each win just over a quarter of them. Indeed, various authors (see e.g. Eaton and Lipsey, 1975; Shaked, 1982; Osborne, 1993; and references therein) have argued that there is no pure (i.e., non-random) Nash equilibrium for the voter model in this case. By contrast, and somewhat similar in spirit to the present paper, Hug (1995) has shown the existence of Nash equilibria for a different model in which parties attempt to maximise their votes (without any negative payoff for losing), but the policy they will later enact is randomly distributed about their intended choice, and voters attempt to minimise a quadratic loss function based on this random distribution. Another modification (Osborne, 1996) has the parties choose their positions *sequentially*, so that party i can base their position on the already-chosen positions of parties $1, 2, \ldots, i-1$. Under that assumption, it is conjectured that just two parties will contest the election, each with position equal to the median m, and the other $n-2$ parties will all stay out. (Osborne conjectured that the first and last parties would be the ones to enter in this case, but that has been disputed when $n = 12$; see de Vries, 2015 and de Vries et al., 2016.)

These considerations are related to *Duverger's Law* (Duverger, 1951; Riker, 1982; Schlesinger and Schlesinger, 2006), which states essentially that typical single-ballot majority vote systems favour the *dualism* of parties, whereby only two (major) parties will contest elections. Empirically, this law is roughly consistent with elections in the United States with two major parties (Democratic and Republican), but less so for elections in Canada and the United Kingdom and other countries. In terms of game theory models, this law is borne out by the $n = 2$ solution above, and by Osborne's sequential system conjecture, but it does not follow in all models and is open to question.

In the present paper, we consider a simple modification of the above voter model, in which each political party i *attempts* to stake out a position $a_i \in \mathbf{R}$, but is actually *perceived* by the voters (due to the imperfections of political advertising, unanticipated events during the election campaign, etc.) to have a position $x_i \in \mathbf{R}$, where x_i is randomly distributed about a_i. (A related "imperfect beliefs" model is considered by Ogden, 2016, but for $n = 2$ parties only.) We shall show (Corollary 1) that under some mild symmetry assumptions, this model has a Nash equilibrium when $n = 3$ in which *all three* parties contest the election, each by attempting to stake out the median position, i.e. with $a_i = m$ for all i, in contrast to Duverger's Law which would imply that only two parties contest the election while one stays out. Along the way, we prove various results (Theorems 1–4) about unimodality of the win probabilities. We also present some explicit computations in specific cases (Section 6), and some counter-examples when our assumptions do not hold (Section 7).

2 Formal Model

Our formal model is as follows. The voters are distributed according to some fixed probability density v on $\mathbf{R}$. Each party has some fixed uncertainty probability distribution G_i. We shall write "$G_i(dx_i - a_i)$" for the distribution G_i shifted over by an amount a_i, so that e.g. $G_i([c, d] - a_i) = G_i([c - a_i, d - a_i])$, etc. The game proceeds as follows:

- Each of n political parties simultaneously chooses an action value $a_i \in \mathbf{R} \cup \{\mathrm{OUT}\}$.
- If $a_i = \mathrm{OUT}$, then also $x_i = \mathrm{OUT}$, so party i does not contest the election, and receives a payoff of zero.
- If $a_i \in \mathbf{R}$, then party i *attempts* to come in at the position a_i. They are then *perceived* by the voters to come in at position $x_i \in \mathbf{R}$, where x_i has probability distribution given by $G_i(dx_i - a_i)$.
- Conditional on the $\{x_i\}$, each party i with $x_i \in \mathbf{R}$ receives a vote share w_i given by

  $$w_i \;=\; \frac{\int_{t \in R_i} v(t)\, dt}{\#\{j : x_j = x_i\}} \,,$$

  where $R_i = \{t \in \mathbf{R} : |t - x_i| \leq |t - x_j| \text{ for all } 1 \leq j \leq n\}$ is the vote region won (or tied for winning) by Party i.
- For each Party i with $x_i \in \mathbf{R}$, their payoff is 1 if they have the highest vote share w_i, or $1/k$ if they tie with a total of k parties for the highest vote share, or -1 if their vote share is strictly less than that of some other party.

3 Assumptions

We shall make the following assumptions.

(A1) The voter density v is symmetric about some fixed central median value $m \in \mathbf{R}$, i.e. $v(m-z) = v(m+z)$ for all $z \in \mathbf{R}$.

(A2) Each uncertainty distribution G_i has probability density g_i which is symmetric about 0, i.e. $g_i(-z) = g_i(z)$ for all $z \in \mathbf{R}$.

(A3) Each party's perceived position x_i is sufficiently close to their attempted position a_i. Specifically, we assume that $|x_i - a_i| \leq M$ (or equivalently g_i is supported on $[-M, M]$) for each i, where the maximal uncertainty constant $M > 0$ is small enough that $\int_{m-M/2}^{m+M/2} v(x)\, dx \leq 1/3$.

For example, these assumptions are all satisfied in the case where v is the Uniform$[0,1]$ density function, with $m = 1/2$ and $n = 3$ and $M = 1/3$.

We shall also sometimes require one of the following (increasingly strong) additional conditions on some of the g_i.

(A4) The uncertainty density g_i is positive in a neighbourhood of 0, i.e. there is $\delta > 0$ with $g(z) > 0$ whenever $|z| < \delta$.

(A4*) The uncertainty density g_i is (weakly) unimodal about 0, i.e. if $m \leq z_1 \leq z_2$ or if $z_2 \leq z_1 \leq m$, then $g_i(z_1) \geq g_i(z_2)$.

(A4)** The uncertainty density g_i is *strongly* unimodal about 0, i.e. if $m \leq z_1 < z_2$ or if $z_2 \leq z_1 \leq m$, then $g_i(z_1) > g_i(z_2)$.

Clearly (A4**) implies (A4*). Furthermore, since the g_i are density functions, it is easily seen that (A4*) implies (A4). So, the conditions (A4) and (A4*) and (A4**) are in increasingly strong order. For a specific example, if g_i is the Uniform$[-0.1, 0.1]$ density, then g_i satisfies (A4) and (A4*) but not (A4**).

4 Results

We have the following results (all proved in the next section).

Theorem 1 *Assume (A1)–(A3), and that we have $n = 3$ parties. Suppose Parties 1 and 2 both attempt to take the central position, i.e. choose $a_1 = a_2 = m$. Then it is (weakly) optimal for Party 3 to be perceived to be in the position $x_3 = m$, i.e. for all $t \in \mathbf{R}$,*

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m, \ x_3 = m] \ \geq \ \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m, \ x_3 = t] \,.$$

Theorem 2 *Assume (A1)–(A3), and that we have $n = 3$ parties, and that (A4*) holds for g_3. Suppose Parties 1 and 2 both attempt to take the central position, i.e. choose $a_1 = a_2 = m$. Then it is (weakly) optimal for Party 3 to attempt to take the position $a_3 = m$, i.e. for all $t \in \mathbf{R}$,*

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m, \ a_3 = m] \ \geq \ \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m, \ a_3 = t] \,.$$

By applying Theorem 2 separately for each of the three parties, we immediately obtain:

Corollary 1 *Assume (A1)–(A3), and that we have $n = 3$ parties, and that (A4*) holds for each uncertainty measure g_i. Then the set of actions $a_1 = a_2 = a_3 = m$ is a Nash equilibrium, i.e. no one party can increase their expected payoff by changing their action while the other two actions remain fixed.*

Corollary 1 contrasts with the results of Hug (1995), who obtains a Nash equilibrium for $n = 3$ parties for a different model, but only upon assuming such things as unequal variances for the different parties. Corollary 1 has no such restrictions, and applies (among others) to the case where each uncertainty density g_i is the same.

Theorems 1 and 2 give only *weak* optimality, i.e. they allow for the possibility that another position is equally good. That is sufficient to prove a Nash equilibrium, as in Corollary 1. But it is possible to get stronger results if we use the stronger assumption (A4**).

Theorem 3 *Assume (A1)–(A3), and that we have $n = 3$ parties, and that (A4) holds for g_1 and g_2. Suppose parties 1 and 2 both attempt to take the central position, i.e. choose*

$a_1 = a_2 = m$. *Then it is strongly optimal for Party 3 to be perceived in the position* $x_3 = m$, *i.e. for all* $t \neq m$,

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ x_3 = m] \ > \ \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ x_3 = t]\,.$$

Theorem 4 *Assume (A1)–(A3), and that we have* $n = 3$ *parties, and that (A4) holds for* g_1 *and* g_2*, and that (A4**) holds for* g_3*. Suppose parties 1 and 2 both attempt to take the central position, i.e. choose* $a_1 = a_2 = m$*. Then it is strongly optimal for Party 3 to attempt to take the position* $a_3 = m$*, i.e. for all* $t \neq m$,

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ a_3 = m] \ > \ \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ a_3 = t]\,.$$

By applying Theorem 4 separately for each of the three parties, we immediately obtain:

Corollary 2 *Assume (A1)–(A3), and that we have* $n = 3$ *parties, and that (A4**) holds for each uncertainty measure* g_i*. Then the set of actions* $a_1 = a_2 = a_3 = m$ *is a strict Nash equilibrium, i.e. if any one party changes their action while the other two actions remain fixed then that will strictly decrease their expected payoff.*

5 Theorem Proofs

We now proceed to prove the theorems. Our proof involves several lemmas. Sometimes we specify the parties' actions a_i (which are then subject to uncertain perception as per the G_i distributions), and sometimes we specify the parties' precisely perceived positions x_i (which are not subject to any uncertainty). Many of the lemmas have a part (a) and a part (b); roughly speaking, each part (a) provides weak inequalities suitable for Theorems 1 and 2, while each part (b) provides strict inequalities under stronger assumptions which are needed for Theorems 3 and 4.

We begin with two simple observations. First, (A3) together with (A1) implies that v is positive on $[m - M, m + M]$ and beyond. Indeed, it follows from (A1) that

$$\int_{m-M/2}^{m+M/2} v(x)\,dx \ \geq \ \int_{m-M}^{m-M/2} v(x)\,dx + \int_{m+M/2}^{m+M} v(x)\,dx\,.$$

If $v(m + z) = 0$ whenever $|z| > M$, then $\int_{m-M}^{m+M} v(x)\,dx = 1$, hence $\int_{m-M/2}^{m+M/2} v(x)\,dx \geq 1/2$, contradicting (A3). Second, (A3) implies that if $x_1, x_2, \ldots, x_n \in (m - M, m + M)$, then the vote share of any party i whose position is between two other parties must be less than 1/3, so for $n \geq 3$ only the biggest or smallest position can obtain the largest vote share.

For our first lemma, write $Q(r,s,t)$ for the probability that Party 3 wins the election (i.e., receives the highest vote share) under the assumptions that the uncertainty distributions are given by $G_1\{-r\} = G_1\{r\} = 1/2$ and $G_2\{-s\} = G_1\{s\} = 1/2$, and the other parties' actions are given by $a_1 = a_2 = m$, and Party 3 comes in at the perceived position $x_3 = m + t$ (with no uncertainty), for fixed $r, s, t > 0$. We have the following fact:

Lemma 1 *Assuming (A1) and (A3), if $0 \leq t_1 < t_2 \leq M$, and $\{t_1, t_2, r, s\}$ are all distinct, then* **(a)** *if $r, s \in (0, M)$, then $Q(r,s,t_1) \geq Q(r,s,t_2)$, and* **(b)** *if $r, s \in (t_1, t_2)$, then $Q(r,s,t_1) > Q(r,s,t_2)$.*

Proof. Assume that $r < s$; the case $r > s$ then follows by symmetry. We wish to compute $Q(r,s,t)$ as a function of t, for fixed r, s. We proceed case by case.

Suppose first that $t < r$. Then if x_1 and x_2 are on the same side of m, then Party 3's winning region R_3 contains everything on the opposite side of m plus more, so since $v(z) > 0$ for $|z| < M$, Party 3 wins more than half the votes, hence the most votes. However, if x_1 and x_2 are on opposite sides of m, then Party 3 is in the middle and hence loses by (A3). Thus, in this case, Party 3 wins if and only if x_1 and x_2 are on the same side of m. Thus, $Q(r,s,t) = 1/2$.

Next, suppose $t > s$. Then Party 3 cannot win if $x_1 > 0$ or $x_2 > 0$. If $x_1 = -r$ and $x_2 = -s$, then Party 3's winning region $R_3 = ((t-r)/2, \infty)$, while Party 2's winning region $R_2 = (-\infty, -(r+s)/2)$. Then, Party 3 wins if and only if $\int_{R_3} v(x)\,dx > \int_{R_2} v(x)\,dx$. Using symmetry and that $v(z) > 0$ for $|z| < M$, this happens if and only if $(t-r)/2 < (r+s)/2$, i.e. $t < s + 2r$. So, $Q(r,s,t) = 1/4$ if $t < s + 2r$, otherwise $Q(r,s,t) = 0$.

Finally, suppose that $r < t < s$. Then if $x_2 = s$, then x_3 is in the middle (whether $x_2 = -r$ or $x_2 = r$), so Party 3 loses by (A3). If $x_2 = -s$ and $x_1 = r$, then $R_3 = ((t+r)/2, \infty)$, while $R_2 = (-\infty, -(s-r)/2, \infty)$, so by the symmetry and positivity of v, Party 3 wins if and only if $(t+r)/2 < (s-r)/2$, i.e. $t < s - 2r$ (which is only possible if $s > 2r$). If $x_2 = -s$ and $x_1 = -r$, then $R_3 = ((t-r)/2, \infty)$, while $R_2 = (-\infty, -(s+r)/2, \infty)$, so again by symmetry and positivity of v, Party 3 wins if and only if $(t-r)/2 < (s+r)/2$, i.e. $t < s + 2r$, which always holds since $t < s$. So here $Q(r,s,t) = 1/2$ if $t < s - 2r$ and $s > 2r$, otherwise $Q(r,s,t) = 1/4$.

In summary, if $0 < r < s < M$ and $2r < s$, then

$$Q(r,s,t) \;=\; \begin{cases} 1/2, & 0 \le t < r \\ 1/2, & r < t < s \\ 1/4, & s < t < s+2r \\ 0, & t > s+2r \end{cases}$$

Or, if instead $0 < r < s < M$ and $2r > s$, then

$$Q(r,s,t) \;=\; \begin{cases} 1/2, & 0 \le t < r \\ 1/4, & r < t < s \\ 1/4, & s < t < s+2r \\ 0, & t > s+2r \end{cases}$$

In either situation, it is easily checked directly that the values of $Q(r,s,t)$ satisfy the stated conclusions in both parts (a) and (b) of the lemma. ∎

Remark. Lemma 1 assumes that $\{r, s, t_1, t_2\}$ are all distinct, thus avoiding complications arising from *ties* for highest voting share. Fortunately we can get away with this, since our theorems assume that the uncertainty distributions have densities and are thus absolutely continuous with respect to Lebesgue measure, so that ties have probability zero.

To continue, let $Y(x_1, x_2, x_3) = 1$ if Party 3 receives the highest vote share when each party i comes in at position x_i, otherwise $Y(x_1, x_2, x_3) = 0$. Then by inspection, our previous quantity $Q(r,s,t)$ can be expressed as

$$\begin{aligned} Q(r,s,t) \;=\; & \frac{1}{4}\Big[Y(m-r, m-s, m+t) + Y(m-r, m+s, m+t) \\ & + Y(m+r, m-s, m+t) + Y(m+r, m+s, m+t)\Big]. \end{aligned} \tag{1}$$

Next, let

$$W(t) \;:=\; \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ x_3 = m+t]$$

be the probability that Party 3 wins, given that Party 3 comes in at the precise position $m+t$, while Parties 1 and 2 attempt to come in at position m. Then by symmetry,

$$W(-t) \;=\; W(t)\,, \quad t \in \mathbf{R}\,. \tag{2}$$

And, by definition, for any uncertainty distributions G_1 and G_2,

$$W(t) \;=\; \int\!\!\int Y(r, s, m+t)\; G_1(dr - m)\, G_2(ds - m)\,,$$

with Y as above. We then have the following.

Lemma 2 *Assuming (A1)–(A3),* **(a)** *if* $0 \le t_1 < t_2 < \infty$*, then* $W(t_1) \ge W(t_2)$*, and* **(b)** *if also* g_1 *and* g_2 *satisfy (A4), then there is* $\delta > 0$ *such that if* $0 \le t_1 < t_2 < \delta$*, then* $W(t_1) > W(t_2)$*.*

Proof. We have that

$$W(t) = \int \int Y(r,s,t)\ G_1(dr-m)\,G_2(ds-m)$$

$$= \int_{x_2} \int_{x_1} Y(x_1, x_2, m+t)\ g_1(x_1-m)\,g_2(x_2-m)\,dx_1\,dx_2$$

$$= \int_{s=-\infty}^{\infty} \int_{r=-\infty}^{\infty} Y(m+r, m+s, m+t)\ g_1(r)\,g_2(s)\,dr\,ds$$

$$= \int_{s=0}^{\infty} \int_{r=0}^{\infty} \big[Y(m-r, m-s, m+t) + Y(m-r, m+s, m+t)$$
$$+Y(m+r, m-s, m+t) + Y(m+r, m+s, m+t)\big]\ g_1(r)\,g_2(s)\,dr\,ds$$

$$= 4\int_{s=0}^{\infty} \int_{r=0}^{\infty} Q(r,s,t)\ g_1(r)\,g_2(s)\,dr\,ds\,, \tag{3}$$

where the last line uses (1). Now, by Lemma 1(a), if $0 \le t_1 < t_2 < \infty$, then $Q(r,s,t_2) \ge Q(r,s,t_1)$ for all $r \ne s$, so by (3), $W(t_1) \ge W(t_2)$, as claimed.

For part (b), by (A4) we can find $\delta > 0$ such that $g_1(z) > 0$ and $g_2(z) > 0$ for $0 < z < \delta$. Then if $0 \le t_1 < t_2 < \delta$, then g_1 and g_2 give positive weight to values of $r, s \in (t_1, t_2)$. Hence, Lemma 1(b) implies that $W(t_1) > W(t_2)$, as claimed. ∎

Proof of Theorem 1. This follows from Lemma 2(a) with $t_1 = 0$, combined with equation (2). ∎

Proof of Theorem 3. This follows from Lemma 2(b) with $t_1 = 0$, combined with equation (2). ∎

To prove Theorems 2 and 4, note that

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ a_3 = t] \ = \ \int W(z)\,G_3(dz-t) \ = \ \int_{z=-\infty}^{\infty} W(z)\,g_3(z-t)\,dz\,. \tag{4}$$

Now, Lemma 2(a)/(b) and (A4*)/(A4**) say that W and g_3 will generally be weakly/strongly unimodal about 0. Hence, we need a general result about maximising integrals of products of unimodal functions. We begin with a brief technical lemma.

Lemma 3 *Let $a, b > 0$, let $J(x) = \mathbf{1}_{|x| \le a}$ and $K(x) = \mathbf{1}_{|x| \le b}$, and let $I(\alpha) = \int J(x)\, K(x + \alpha)\, dx$. Then* **(a)** *$I(\alpha) \le I(0)$ for all $\alpha \in \mathbf{R}$. Furthermore,* **(b)** *if $|\alpha| > a + b$, then $I(\alpha) = 0$.*

Proof. Assume without loss of generality that $a \ge b$, and $\alpha \ge 0$. We compute that $J(x)\, K(x + \alpha) = \mathbf{1}_{-b+\alpha \le x \le \min(a, b+\alpha)}$, so $I(\alpha) = \max\big(0,\ \min(a, b + \alpha) - (-b + \alpha)\big) = \max\big(0,\ \min(a + b - \alpha,\, 2b)\big)$. This is a non-increasing function of $\alpha \ge 0$, thus giving the result. Furthermore, if $\alpha > a + b$, then $\min(a + b - \alpha,\, 2b) = 0$, so $I(\alpha) = 0$. ∎

We next use Lemma 3 to prove the result about integrals of products of unimodal functions.

Lemma 4 *Suppose f and g are two non-negative integrable functions. Assume f and g are symmetric about a common mode $m \in \mathbf{R}$, i.e. $f(m - z) = f(m + z)$ and $g(m - z) = g(m + z)$ for all $z \in \mathbf{R}$. Also assume f and g are (weakly) unimodal about m, i.e. for $0 \le t_1 \le t_2 < \infty$, $f(m + t_1) \ge f(m + t_2)$ and $g(m + t_1) \ge g(m + t_2)$. Let $I(\alpha) = \int f(x)\, g(x + \alpha)\, dx$. Then* **(a)** *$I$ is (weakly) maximised at $\alpha = 0$, i.e. for all $\alpha \in \mathbf{R}$, $I(0) \ge I(\alpha)$. Also* **(b)** *if in addition f and g are both* strongly *unimodal in a neighbourhood of m, i.e. for some $\delta > 0$, $f(m + s) < f(m + r)$ and $g(m + s) < g(m + r)$ whenever $0 \le r < s \le \delta$, then $I(0) > I(\alpha)$.*

Proof. For each $n \in \mathbf{N}$, approximate g from below by simple functions, as follows. For $i = 1, 2, 3, \ldots$, let $u_{n,i} = \inf\{f(x) : |x - m| \le i2^{-n}\}$, and let $\beta_{n,i} = u_{n,i} - u_{n,i+1} \ge 0$. Let $J_{n,i}(x) = \mathbf{1}_{|x-m| \le i2^{-n}}$. Then let $f_n(x) = \sum_{i=1}^{\infty} \beta_{n,i}\, J_{n,i}(x)$. Similarly let $v_{n,i} = \inf\{g(x) : |x - m| \le i2^{-n}\}$, and $\gamma_{n,i} = v_{n,i} - v_{n,i+1} \ge 0$, and $g_n(x) = \sum_{i=1}^{\infty} \gamma_{n,i}\, J_{n,i}(x)$. Then by the Monotone Convergence Theorem, $I(\alpha) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta_{n,i}\gamma_{n,j} \int J_{n,i}(x)\, J_{n,j}(x + \alpha)\, dx$, where $\beta_{n,i}\gamma_{n,j} \ge 0$. Hence, part (a) follows from Lemma 3(a).

For part (b), let $\alpha > 0$, and let $d = f(m + \min(\delta, \alpha/2))$ and $e = g(m + \min(\delta, \alpha/2))$, and $d' = f(m + \frac{1}{2}\min(\delta, \alpha/2))$ and $e' = g(m + \frac{1}{2}\min(\delta, \alpha/2))$. Then by the strong unimodality, $f(m) > d' > d$ and $g(m) > e' > e$, but $f(x) \le d$ and $g(x) \le e$ whenever $|x - m| \ge \alpha/2$. Now, for any $x \in \mathbf{R}$, either $|x - m| \ge \alpha/2$ or $|x + \alpha - m| \ge \alpha/2$ or both. Thus, for all

$x \in \mathbf{R}$, either $f(x) \le d$ or $g(x+\alpha) \le e$ or both, so the product $f(x)\,g(x)$ can never be more than $\max[d\,g(x),\ e\,f(x)]$. It follows from Lemma 3(b) that in the above expansion, $\beta_{n,i}\gamma_{n,j}\int J_{n,i}(x)\,J_{n,j}(x+\alpha)\,dx = 0$ whenever $i2^{-n} < \alpha/2$ and $j2^{-n} < \alpha/2$. Furthermore by Lemma 3(a) again, for *every* term of the expansion, $\beta_{n,i}\gamma_{n,j}\int J_{n,i}(x)\,J_{n,j}(x+\alpha)\,dx \le \beta_{n,i}\gamma_{n,j}\int J_{n,i}(x)\,J_{n,j}(x)\,dx$. Hence, if we let $h(x) = f(x)\,g(x)$ when $f(x) < d$ or $g(x) < e$ or both, and $h(x) = \max[d\,g(x),\ e\,f(x)]$ otherwise, then from the above expansion, $I(\alpha) \le \int h(x)\,dx$. But for $|x| \le \frac{1}{2}\min(\delta, \alpha/2)$, we have that $f(x) \ge d$ and $g(x) \ge e$, so $h(x) = \max[d\,g(x),\ e\,f(x)]$, whence

$$f(x)\,g(x) - h(x) = f(x)\,g(x) - \max[d\,g(x),\ e\,f(x)] = \min[f(x)\,g(x) - d\,g(x),\ f(x)\,g(x) - e\,f(x)]$$
$$= \min[(f(x)-d)\,g(x),\ f(x)\,(g(x)-e)] \ge \min[(d'-d)\,e',\ d'\,(e'-e)]\,.$$

We then compute that

$$I(0) - I(\alpha) \;\ge\; \int_{x=-\infty}^{\infty} f(x)\,g(x)\,dx \;-\; \int_{x=-\infty}^{\infty} h(x)\,dx \;=\; \int_{x=-\infty}^{\infty} [f(x)\,g(x) - h(x)]\,dx$$
$$\ge \int_{x=-\frac{1}{2}\min(\delta,\alpha/2)}^{\frac{1}{2}\min(\delta,\alpha/2)} [f(x)\,g(x) - h(x)]\,dx \;\ge\; \int_{x=-\frac{1}{2}\min(\delta,\alpha/2)}^{\frac{1}{2}\min(\delta,\alpha/2)} \min[(d'-d)\,e',\ d'\,(e'-e)]\,dx$$
$$= \;\min(\delta, \alpha/2)\;\min[(d'-d)\,e',\ d'\,(e'-e)] \;>\; 0\,,$$

as claimed. ∎

Remark. If g is a C^1 function, then it is tempting to try to prove Lemma 4 by differentiating under the integral sign and using symmetry, viz.

$$I'(\alpha) \;=\; \int_{-\infty}^{\infty} f(x)\,g'(x+\alpha)\,dx \;=\; \int_{-\infty}^{m} [f(x)\,g'(x+\alpha) + f(2m-x)\,g'(2m-x+\alpha)]\,dx$$
$$= \int_{-\infty}^{m} f(x)\,[g'(x+\alpha) + g'(2m-x+\alpha)]\,dx\,.$$

Now, if $\alpha > 0$ and $x \le m$, then $|m-(2m-x+\alpha)| = |m-(x-\alpha)| > |m-(x+\alpha)|$, i.e. $x+\alpha$ is closer to the mode m than $2m-x-\alpha$ is. This suggests that perhaps $g'(x+\alpha)+g'(2m-x+\alpha) < 0$, which would give the result. However, in fact this inequality need not be true, and it is not clear how to complete a proof in this manner (even assuming that g is C^1).

Proof of Theorem 2. This follows by combining Lemma 2(a) and Lemma 4(a) with (4). ∎

Proof of Theorem 4. This follows by combining Lemma 2(b) and Lemma 4(b) with (4). ∎

6 Explicit Computations

To make the previous theoretical results more concrete, we now do some explicit computations. Let v be the Uniform[0,1] voter density, with $m = 1/2$ and $M = 1/3$. Let

$$\text{WinProb}(z) = \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\, x_3 = z]$$

be the probability that Party 3 receives the most votes if they come in at the (definite) position $x_3 = z$, while Parties 1 and 2 attempt to come in at $a_1 = a_2 = m$, and are then subject to an uncertainty density $g_1 = g_2 = g$ for various choices of g which are supported on $[m - M, m + M] = [1/6, 5/6]$, and which are symmetric and unimodal about $m = 1/2$. (Thus, $\text{WinProb}(z) = W(z - m)$ with $W(t)$ as before.)

Theorems 1 and 3 above state in this case that, if the assumptions are satisfied, then $\text{WinProb}(z)$ should be symmetric and unimodal about $m = 1/2$. We now proceed to verify that in various specific examples.

6.1 Preliminary Computations

Under the above circumstances, Party 3 (which comes in at z) wins if and only if one of the following mutually exclusive situations arises (where for simplicity we write x for x_1, and y for x_2, and z for x_3):

1. $x < y < z$, and $1 - \frac{y+z}{2} > \frac{x+y}{2}$.
2. $y < x < z$, and $1 - \frac{x+z}{2} > \frac{x+y}{2}$.
3. $z < y < x$, and $\frac{y+z}{2} > 1 - \frac{x+y}{2}$.
4. $z < x < y$, and $\frac{x+z}{2} > 1 - \frac{x+y}{2}$.

For fixed x and y, let $p_i(z)$ be the probability of situation i above, for $i = 1, 2, 3, 4$. Then clearly $p_2(z) = p_1(z)$ and $p_4(z) = p_3(z)$. Furthermore $p_3(z) = p_1(1 - z)$. Hence,

$$\text{WinProb}(z) \;:=\; = \; p_1(z) + p_2(z) + p_3(z) + p_4(z) \;=\; 2\,p_1(z) + 2\,p_1(1 - z)\,. \qquad (5)$$

So, to compute $\text{WinProb}(z)$, it suffices to compute $p_1(z)$ for all $z \in [1/6, 5/6]$.

The assumptions and conditions for situation 1 above imply that $x < y < z$, and $y <$

$1-(x+z)/2$, and also $x < 2-2y-z < 2-2x-z$ whence $x < \frac{2-z}{3}$. Hence,

$$p_1(z) \;=\; \int_{x=0}^{\min(z,(2-z)/3)} g(x-1/2) \int_{y=x}^{\min(z,1-(x+z)/2)} g(y-1/2)\; dy\; dx \tag{6}$$

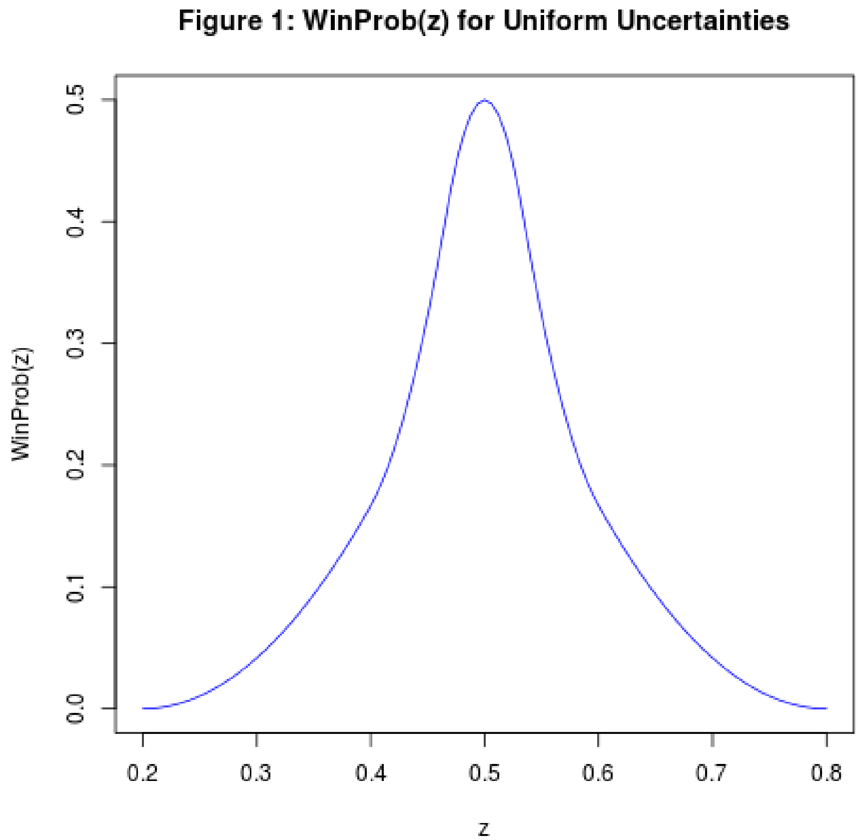
Formulae (5) and (6) then give, in principle, an expression for WinProb(z). We consider three specific cases.

6.2 Uniform Uncertainties

Suppose first that g corresponds to the Uniform$[-0.1, 0.1]$ density, i.e. $g(x) \equiv 5\,\mathbf{1}_{2/5 \le x \le 3/5}$, so that Parties 1 and 2 have positions which are Uniform$[2/5, 3/5]$. In this case, after considerable effort (with symbolic algebra assistance from the *Mathematica* computation system, Wolfram 1988), we compute that

$$\mathrm{WinProb}(z) \;=\; \begin{cases} 0, & z \le 1/5 \\ \frac{1}{6}(1-5z)^2, & 1/5 \le z \le 2/5 \\ \frac{5}{6}(5-26z+35z^2), & 2/5 \le z \le 7/15 \\ \frac{1}{3}(-61+250z-250z^2), & 7/15 \le z \le 8/15 \\ \frac{5}{6}(14-44z+35z^2), & 8/15 \le z \le 3/5 \\ \frac{1}{6}(4-5z)^2, & 3/5 \le z \le 4/5 \\ 0, & z \ge 4/5 \end{cases}$$

This function is graphed in Figure 1. As can be seen from the graph, the function is indeed symmetric and (strongly) unimodal about $m = 1/2$, consistent with Theorems 1 and 3.

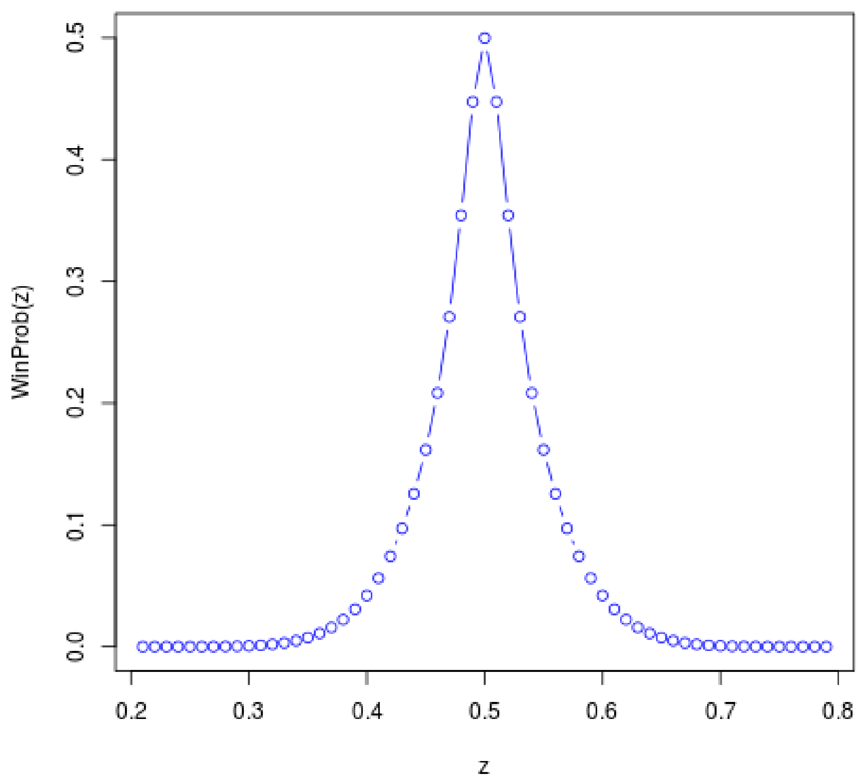


6.3 Quadratic Uncertainties

Suppose instead that $g(x) = 1500(0.1 - |x|)^2 \mathbf{1}_{-0.1 \leq x \leq 0.1}$ is a *quadratic* density function on $[-0.1, 0.1]$.

In this case, it appears quite challenging to compute WinProb(z) exactly from the formulae (5) and (6). But it is more straightforward to do *numerical* calculations to illustrate the values of WinProb(z). The results are shown in Figure 2. As can be seen from the figure, the function does indeed appear to be symmetric and (strongly) unimodal about $m = 1/2$, as it must be by Theorems 1 and 3.

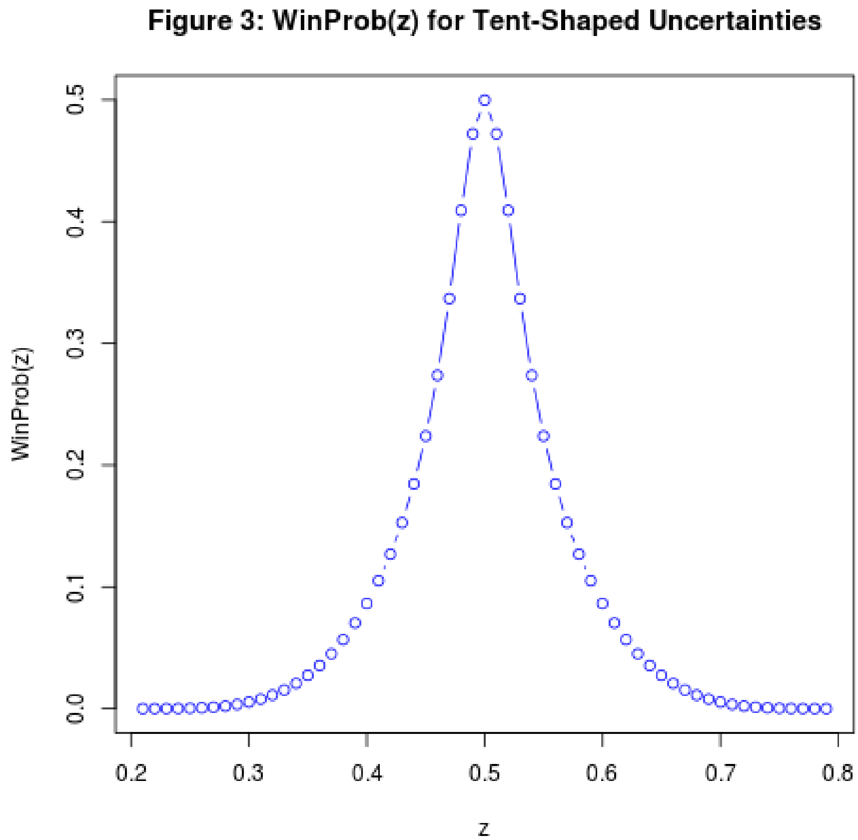
Figure 2: WinProb(z) for Quadratic Uncertainties



6.4 Tent-Shaped Uncertainties

Finally, suppose that $g(x) = 100(0.1 - |x|)\mathbf{1}_{-0.1 \leq x \leq 0.1}$ is a "tent-shaped" density function on $[-0.1, 0.1]$.

In this case, it again appears quite challenging to compute $\mathrm{WinProb}(z)$ exactly from (5) and (6). But again it is straightforward to do *numerical* calculations of $\mathrm{WinProb}(z)$. The results are shown in Figure 3. As can be seen from the figure, the function again appears to be symmetric and (strongly) unimodal about $m = 1/2$, as per Theorems 1 and 3.



7 Counter-Examples

We next consider a few simple counter-examples if our assumptions are violated.

7.1 Asymmetric Counter-Examples

We first illustrate that if the uncertainty densities g_i are not required to be *symmetric*, then our results may be false (even if each g_i has mean zero). We again assume that v is the Uniform[0,1] density, with median $m = 1/2 = 0.5$.

For a first illustration, suppose the uncertainty is such that voters perceive each party's position as 0.01 lower with probability 9/10, or 0.09 higher with probability 1/10 (so the uncertainty has mean zero). Suppose Parties 1 and 2 attempt to come in at $a_1 = a_2 = m = 0.5$, and Party 3 attempts to come in at $a_3 = 0.51$. Then Party 3 will win if all three parties' perceived positions are 0.01 lower (since then Parties 1 and 2 will be perceived at

0.49, while Party 3 will be perceived at 0.50). Hence, Party 3 will win with probability at least $(9/10)^3 = 0.729 > 1/3$. It follows that Party 3 would prefer to attempt to come in at 0.51 than at 0.50 (where they would have probability 1/3 of winning, by symmetry). Hence, in this case, it is *not* an equilibrium to have all three parties attempt to come in at 0.5.

Now, the uncertainty densities in this first illustration are not unimodal. However, they can be modified to be unimodal, while remaining mean-zero, while still leading to essentially the same conclusions as before. Indeed, let

$$g(x) \;=\; \begin{cases} 0, & x < -0.01 \\ 90, & -0.01 \le x < 0 \\ 10/9, & 0 \le x < 0.09 \\ 0, & x \ge 0.1 \end{cases}$$

Then $\int g(x)\,dx = 90(0.01) + (10/9)(0.09) = 1$, and $g \ge 0$, so g is a valid density function. Also $\int x\,g(x)\,dx = 90(0.01)(-0.005) + (10/9)(0.09)(0.045) = 0$, so g has mean 0. Also, g is (weakly) unimodal around 0 by inspection.

Suppose again that Parties 1 and 2 attempt to come in at $a_1 = a_2 = m = 0.5$, and Party 3 attempts to come in at $a_3 = 0.51$. Then, regarding g as a mixture of two uniform distributions, we can say that with probability $(0.9)^3 = 0.729$, x_1 and x_2 are each distributed (independently) as Uniform$[0.49, 0.5]$, while x_3 is distributed as Uniform$[0.5, 0.51]$. In this case, Party 3 receives vote share equal to $1 - (x_3 + \max(x_1, x_2))/2$, while whichever of Party 1 and 2 has a lower position receives vote share of $(x_1 + x_2)/2$. So, Party 3 wins if $1 - (x_3 + \max(x_1, x_2))/2 > (x_1 + x_2)/2$, i.e. $2 - x_3 - \max(x_1, x_2) > x_1 + x_2$. This is computed to have probability 8/9. Thus, by attempting to come in at $a_3 = 0.51$, Party 3 wins with probability at least $(0.9)^3(8/9) = 81/125 = 0.648$, which is still much greater than the win probability 1/3 that Party 3 would obtain (by symmetry) by attempting to come in at 0.5.

We conclude that if the uncertainty densities are required to be unimodal and mean zero but not symmetric, then our theorems are false, and in general it is *not* a Nash equilibrium for all three parties to attempt to come in at $a_i = m$.

7.2 A Large-n Counter-Example

Suppose our assumptions hold, but n is larger than 3. Do our theorems still hold in that case? The answer to this question is also no in general.

For example, let v again be the Uniform[0,1] density (so that (A1) is satisfied with $m = 1/2$), let the number of parties be n, and let the common uncertainty density g be symmetric about 0. Suppose g has the properties that if $X \sim g$, then $\mathbf{P}(-1/n^2 \leq X \leq 0) \geq 1/4$, and $\mathbf{P}(X \geq 0.1) = \mathbf{P}(X \leq -0.1) = 1/n$, and $\mathbf{P}(X \geq 0.1 - 1/n^2) \leq 1/n + 1/n^2$, and that (A2) and (A3) and (A4**) are satisfied, as can easily be arranged by adjusting the density g appropriately.

Suppose for this example that parties $1, 2, \ldots, n-1$ all attempt to come in at $a_i = m = 1/2$. Then if Party n also attempts to come in at $a_n = m = 1/2$, then by symmetry they will win with probability $1/n$. But suppose instead that Party n attempts to come in at $a_n = 0.6$. We claim that, for large enough n at least, this will give Party n a win probability which is larger than $1/n$.

Indeed, suppose it happens that Party n has a perceived position $x_n \in [0.6 - 1/n^2,\ 0.6]$, and precisely two distinct parties i and j have actual perceived positions $x_i, x_j \leq 0.4$, and all the other parties' actual perceived positions satisfy $x_i \in (0.4,\ 0.6 - 1/n^2)$, If so, then Party n will have a win region $R_n \supseteq [0.6, 1]$ and hence a vote share of at least 0.4, while all other parties' vote shares will be < 0.4, so Party n will win.

But this event has probability $\geq (1/4)\binom{n-1}{2}(1/n)^2(1 - 2/n - 1/n^2)^{n-3}$. This probability equals 0.0205 when $n = 4$, or 0.0188 when $n = 5$, or 0.0181 when $n = 6$, etc. More importantly, as $n \to \infty$, this probability converges to $(1/8)e^{-2} \doteq 0.0169$. So, for all sufficiently large n, this probability is greater than $1/n$. This means that for large enough n, if Parties $1, 2, \ldots, n-1$ all attempt to come in at $a_i = m = 1/2$, then Party n would prefer to attempt to come in at $a_n = 0.6$ than at $a_n = m = 1/2$.

This demonstrates that our Corollary 1 and the other theorems proved herein for $n = 3$ do not hold for sufficiently large n.

7.3 The Necessity of the Condition (A3)

Our assumption (A3) is that the perceived positions are not *too far* from the attempted positions, which seems reasonable. Nevertheless, we did wonder if it might be possible to prove our theorems without assuming (A3). However, after much effort, we concluded that this is impossible, as the following simple counter-examples show.

Let v be the Uniform$[-1, 1]$ density (so $m = 0$), with $a_1 = a_2 = 0$. Suppose first that the uncertainties are such that $x_1 = -1$ or $+1$ with probability 1/2 each, and $x_2 = -0.4$ or $+0.4$ with probability 1/2 each. Then if $x_3 = 0$, then Party 3 wins if and only if x_1 and x_2 have the same sign, thus with probability 1/2. But if $x_3 = 0.2$, then Party 3 wins in those cases and also if $x_1 = -1$ and $x_2 = +0.4$ (since in that case Party 3's winning region is $[-0.6, 0.3]$ with vote share 0.45, while Party 1's winning region is $[-1, -0.6]$ with vote share 0.2, and Party 2's winning region is $[0.3, 1]$ with vote share 0.35), so Party 3's win probability increases from 1/2 to 3/4, contradicting the conclusions of each of Theorems 1, 2, 3, and 4 in this case.

Now, these uncertainty measures do not satisfy our other assumptions either. However, the counter-example can be modified so they do. Specifically, let g_1 be symmetric and supported on $[-1.01, -0.99] \cup [-0.01, 0.01] \cup [0.99, 1.01]$, let g_2 be symmetric and supported on $[-0.41, -0.39] \cup [-0.01, 0.01] \cup [0.39, 0.41]$, and let g_3 be symmetric and supported on $[-0.01, 0.01]$, and with with $\int_{-0.01}^{0.01} g_i(x)\, dx \leq 0.01$ for $i = 1, 2$ (so the amount of mass of g_1 and g_2 near zero is very small). Then it is easily checked that if $|x_3| \leq 0.01$, then at least for $|x_1| > 0.01$ and $|x_2 > 0.01$, Party 3 wins if and only if x_1 and x_2 have the same sign, hence with probability within 0.02 of 1/2. But if $1.99 < x_3 < 2.01$, then at least for $|x_1| > 0.01$ and $|x_2 > 0.01$, Party 3 wins if x_1 and x_2 have the same sign, and *also* if $x_1 < 0$ and $x_2 > 0$, hence with probability within 0.02 of 3/4. It follows that we still have

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = 0,\ x_3 = 0] \ < \ \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ x_3 = 0.2]\,,$$

and also

$$\mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = 0,\ a_3 = 0] \ < \ \mathbf{P}[\text{Party 3 wins} \,|\, a_1 = a_2 = m,\ a_3 = 0.2]\,.$$

Now, this example certainly satisfies (A1) and (A2). Furthermore, by adjusting the forms of the g_i appropriately within their support intervals, we can ensure that g_1 and g_2 satisfy (A4), and that g_3 satisfies (A4**). However, g_1 does not satisfy (A3) which would require that $|x_i - a_i| \leq M := 2/3$. Hence, this shows that Theorems 1, 2, 3, and 4 all fail if the single assumption (A3) is omitted.

Furthermore, this example still satisfies $|x_i - a_i| \leq 1$, i.e. it satisfies (A3) but with $M = 2/3$ replaced by the constant 1. Indeed, for any $c > 0$, if we adjust the counter-example to instead let v be Uniform$[-c, c]$, and divide each of the above position values by c, then the counter-example still holds exactly as before, but now with $|x_i - a_i| \leq c$, i.e. satisfying (A3) with M replaced by c. This shows that it is not possible to replace the uncertainty bound M in (A3) by *any* fixed value $c > 0$ which does not depend on v. Rather, it is necessary that the bound M be specified in terms of the scale implied by v (which is, indeed, the only meaningful scale available to us in this model). On the other hand, it is *not* possible to make this g_1 and g_2 satisfy (A4*) or (A4**), so this example does *not* demonstrate that (A3) is required to prove Corollaries 1 and 2, though any proof without assuming (A3) would have to be very different in nature.

It is worth considering where our above proofs break down if we do not assume (A3). Consider specifically the proof of Lemma 1 in the case where $0 < t < r < s$. Even without (A3), if x_1 and x_2 are on the same side of m (which has probability 1/2), then as before Party 3 wins more than half the votes, hence the most votes. But what if they are on opposite sides?

Suppose first that $x_1 = m - r$ and $x_2 = m + s$. Then Party 3 receives $\int_{m+(t-r)/2}^{m+(t+s)/2} v(x)dx$, which by (A1) is a weakly *decreasing* function of t for $0 < t < r < s$. Also Party 1 receives $\int_{-\infty}^{m+(t-r)/2} v(x)dx$, which is a weakly *increasing* function of t for $r < s$ by inspection. Also Party 2 receives less than Party 1 since $r < s$. Hence, Party 3 wins if and only if $\int_{m+(t-r)/2}^{m+(t+s)/2} v(x)dx > \int_{-\infty}^{m+(t-r)/2} v(x)dx$, i.e. if and only if $J_{r,s}(t) := \int_{m+(t-r)/2}^{m+(t+s)/2} v(x)dx - \int_{-\infty}^{m+(t-r)/2} v(x)dx > 0$. Since $J_{r,s}(t)$ is a weakly decreasing function of t, Party 3 is *less* likely to win in this way as t increases. Hence, this additional term only re-enforces the inequalities Lemma 1, and does not cause any problems with the proof.

On the other hand, if $x_1 = m+r$ and $x_2 = m-s$, then Party 3 receives $\int_{m+(t-s)/2}^{m+(t+r)/2} v(x)dx$. But this is now a weakly *increasing* function of t for $0 < t < r < s$, not decreasing. This is where our proof breaks down without assuming (A3), making the above counter-example possible.

Acknowledgements. I thank Martin J. Osborne for introducing me to this topic and for many helpful discussions. This research was partially supported by NSERC of Canada.

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