A Lemma About Unimodal Limits

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I thought I needed the following lemma for a paper [1], but it turned out I didn't need it after all. Nevertheless, the proof is simple and kind of cute, so I am posting it here for any interested persons.

LEMMA: Let $g : \mathbf{R} \to \mathbf{R}$ be a non-negative function which is symmetric and unimodal around a mode $m \in \mathbf{R}$. Then there is a sequence $\{g_n\}$ of non-negative C^1 functions which are each symmetric and unimodal about m, and which increase monotonically to g almost everywhere.

PROOF: Let $S : [0,1] \to [0,1]$ by $S(x) = 6\int_0^1 x(1-x)dx = 3x^2 - 2x^3$, so that S has an "S-shaped" graph, and S is C^1 and is strictly increasing on $[0,1]$ with $S(0) = 0$ and $S(1) = 1$ and $S'(0) = S'(1) = 0$. Let $u_{n,i} = \inf_{x \in [(i-1)2^{-n}, i2^{-n}]} g(x)$. Then construct g_n by joining together various S functions on intervals $[i2^{-n}, (i+1)2^{-n})$, by $g_n(x) = u_{n,i} + u_{n,i+1} S(x - i2^{-n})$ for $x \in [m + i2^{-n}, m + (i+1)2^{-n})$ for $i = \ldots, -3, -2, -1$, and $g_n(x) = u_{n,i+1} + u_{n,i+2} S(x - i2^{-n})$ for $x \in [m + i2^{-n}, m + (i+1)2^{-n})$ for $i = 0, 1, 2, \ldots$. Then it is checked that $\{g_n\}$ is symmetric and unimodal about m, and furthermore that $\{g_n(x)\} \nearrow g(x)$ at all continuity points x of g, which must be almost everywhere by the unimodality of g. ∎

[1] J.S. Rosenthal (2016), Nash Equilibria for Voter Models with Randomly Perceived Positions. Available at: `www.probability.ca/jeff/research.html`