A Lemma About Unimodal Limits

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I thought I needed the following lemma for a paper [1], but it turned out I didn’t need it after all. Nevertheless, the proof is simple and kind of cute, so I am posting it here for any interested persons.

**Lemma:** Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a non-negative function which is symmetric and unimodal around a mode \( m \in \mathbb{R} \). Then there is a sequence \( \{g_n\} \) of non-negative \( C^1 \) functions which are each symmetric and unimodal about \( m \), and which increase monotonically to \( g \) almost everywhere.

**Proof:** Let \( S : [0, 1] \rightarrow [0, 1] \) by \( S(x) = 6 \int_0^1 x(1-x)dx = 3x^2 - 2x^3 \), so that \( S \) has an “S-shaped” graph, and \( S \) is \( C^1 \) and is strictly increasing on \([0, 1]\) with \( S(0) = 0 \) and \( S(1) = 1 \) and \( S'(0) = S'(1) = 0 \). Let \( u_{n,i} = \inf_{x \in [(i-1)2^{-n},i2^{-n}]} g(x) \). Then construct \( g_n \) by joining together various \( S \) functions on intervals \([i2^{-n},(i+1)2^{-n})\), by \( g_n(x) = u_{n,i} + u_{n,i+1}S(x-i2^{-n}) \) for \( x \in [m+i2^{-n},m+(i+1)2^{-n}) \) for \( i = \ldots, -3, -2, -1 \), and \( g_n(x) = u_{n,i+1} + u_{n,i+2}S(x-i2^{-n}) \) for \( x \in [m+i2^{-n},m+(i+1)2^{-n}) \) for \( i = 0, 1, 2, \ldots \). Then it is checked that \( \{g_n\} \) is symmetric and unimodal about \( m \), and furthermore that \( \{g_n(x)\} \nearrow g(x) \) at all continuity points \( x \) of \( g \), which must be almost everywhere by the unimodality of \( g \). ■