Mean Squared Error of Variance Estimators

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Let $x_1, x_2, \ldots, x_n$ be i.i.d., each with finite mean m and variance $v = \mathbf{E}[(x_i - m)^2]$ and fourth central moment $w = \mathbf{E}[(x_i - m)^4]$.

Let $B = \frac{1}{a}\sum_{i=1}^{n}(x_i - \overline{x})^2$ be an estimator of v, for some fixed $a > 0$.

We are interested (because of the short paper [3]) in the Mean Squared Error (MSE) when B is used as an estimator for the variance v. A formula is claimed in [4], and related calculations appear in e.g. [1, 2]. Here for completeness we derive a formula for $MSE(B)$, making use of some related moment calculations by Cramér [1].

I thank Mike Evans for helpful suggestions.

Proposition. The MSE of B as an estimator for v is given by

$$MSE(B) \ := \ \mathbf{E}[(B-v)^2] \ = \ \frac{n-1}{na^2}\Big[(n-1)\gamma + n^2 + n\Big]v^2 - \Big[\frac{2(n-1)}{a} - 1\Big]v^2 \,,$$

where $\gamma = \frac{w}{v^2} - 3$ the excess kurtosis (so $w = v^2(\gamma + 3)$).

Proof. Let $\mu_\nu = \mathbf{E}[(x_i - m)^\nu]$, and let $m_\nu = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^\nu$. Then our estimator is $B = \frac{1}{a}\sum_{i=1}^{n}(x_i - \overline{x})^2 = (n/a)m_2$. We want the MSE of B, i.e.

$$MSE(B) \ := \ \mathbf{E}[(B-v)^2] \ = \ \mathbf{E}[B^2] - 2\,v\,\mathbf{E}[B] + v^2$$

Now, Cramér equation (27.4.1) on page 347 says that $\mathbf{E}[m_2] = \frac{n-1}{n}\mu_2$ where $\mu_2 = \mathbf{E}[(x_i - m)^2] = v$. Hence, $\mathbf{E}[B] = (n/a)\mathbf{E}[m_2] = \frac{n-1}{a}v$. Also, Cramér equation (27.4.2) on page 348 says that

$$\mathbf{E}[(m_2)^2] \ = \ \mu_2^2 + \frac{\mu_4 - 3\mu_2^2}{n} - \frac{2\mu_4 - 5\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}\,,$$

where $\mu_2 = v$ as above, and where $\mu_4 = \mathbf{E}[(x_i - m)^4] = w$. Hence,

$$\mathbf{E}[B^2] \ = \ (n/a)^2\,\mathbf{E}[(m_2)^2] \ = \ (n/a)^2\left[\mu_2^2 + \frac{\mu_4 - 3\mu_2^2}{n} - \frac{2\mu_4 - 5\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}\right].$$

Putting this all together,

$$MSE(B) \ = \ \mathbf{E}[B^2] - 2\,v\,\mathbf{E}[B] + v^2$$

$$= (n/a)^2\Big[\mu_2^2 + \frac{\mu_4 - 3\mu_2^2}{n} - \frac{2\mu_4 - 5\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}\Big] - 2v\frac{n-1}{a}v + v^2 .$$

Substituting in $\mu_2 = v$ and $\mu_4 = w = v^2(\gamma + 3)$, this becomes

$$MSE(B) = (n/a)^2\Big[v^2 + \frac{v^2(\gamma+3) - 3v^2}{n} - \frac{2v^2(\gamma+3) - 5v^2}{n^2} + \frac{v^2(\gamma+3) - 3v^2}{n^3}\Big] - 2v\frac{n-1}{a}v + v^2$$

$$= (nv/a)^2\Big[1 + \frac{\gamma}{n} - \frac{2\gamma+1}{n^2} + \frac{v^2\gamma}{n^3}\Big] - \Big[\frac{2(n-1)}{a} - 1\Big]v^2$$

$$= (nv/a)^2\Big[(1 + \frac{1}{n^2}) + \gamma(\frac{1}{n} - \frac{2}{n^2} + \frac{1}{n^3})\Big] - \Big[\frac{2(n-1)}{a} - 1\Big]v^2$$

$$= (nv/a)^2\Big[\frac{n^2+1}{n^2} + \gamma\frac{n^2 - 2n + 1}{n^3}\Big] - \Big[\frac{2(n-1)}{a} - 1\Big]v^2$$

$$= (v/a)^2\Big[(n+1)(n-1) + \gamma\frac{(n-1)^2}{n}\Big] - \Big[\frac{2(n-1)}{a} - 1\Big]v^2$$

$$= \frac{n-1}{na^2}\Big[n(n+1) + \gamma(n-1)\Big]v^2 - \Big[\frac{2(n-1)}{a} - 1\Big]v^2$$

$$= \frac{n-1}{na^2}\Big[(n-1)\gamma + n^2 + n\Big]v^2 - \Big[\frac{2(n-1)}{a} - 1\Big]v^2 . \qquad Q.E.D.$$

References

[1] H. Cramér (1946), Mathematical Methods of Statistics. Princeton University Press.

[2] A. Mood, F. Graybill, and D. Boes (1974), Introduction to the Theory of Statistics (3rd ed.), p. 229. McGraw-Hill, New York City.

[3] J.S. Rosenthal (2015), The kids are alright: divide by n when estimating variance. IMS Bulletin, to appear.

[4] Wikipedia, Mean squared error: Variance. Retrieved August 26, 2015. Available at:
`en.wikipedia.org/wiki/Mean_squared_error#Variance`