STA 261S, Winter 2004, Test #1

(February 11, 2004. Duration: 100 minutes.)

SOLUTIONS

1. Let $\Omega = S = [0,1]$, and let $L_0(\theta \,|\, s) = e^{\theta}$. Determine (with explanation) whether or not each of the following likelihood functions is <u>equivalent</u> to the likelihood function $L_0(\theta \,|\, s)$.

(a) $L_1(\theta \,|\, s) = s^2 + e^{\theta}$.

Solution. *Here* $\frac{L_1(\theta \,|\, s)}{L_0(\theta \,|\, s)} = \frac{s^2+e^{\theta}}{e^{\theta}} = s^2\, e^{-\theta} + 1$, *which depends on* θ. *Hence,* L_1 *is <u>NOT</u> equivalent to* L_0.

(b) $L_2(\theta \,|\, s) = e^{s^2+\theta}$.

Solution. *Here* $\frac{L_2(\theta \,|\, s)}{L_0(\theta \,|\, s)} = \frac{e^{s^2+\theta}}{e^{\theta}} = e^{s^2}$, *which does not depend on* θ. *Hence,* L_2 *<u>IS</u> equivalent to* L_0.

(c) $L_3(\theta \,|\, s) = e^{s^2\theta}$.

Solution. *Here* $\frac{L_3(\theta \,|\, s)}{L_0(\theta \,|\, s)} = \frac{e^{s^2\theta}}{e^{\theta}} = e^{(s^2-1)\theta}$, *which depends on* θ. *Hence,* L_3 *is <u>NOT</u> equivalent to* L_0.

2. Let $\Omega = S = (0,1)$. Suppose the likelihood function, given an observation $s \in S$, is given by $L(\theta \,|\, s) = \theta^{2s}(1-\theta)^{4s}$, for $\theta \in \Omega$.

(a) Compute (with explanation) the Score Function for this likelihood.

Solution. *Here* $\ell(\theta \,|\, s) = \log L(\theta \,|\, s) = 2s\log(\theta) + 4s\log(1-\theta)$, *so the Score Function is* $S(\theta \,|\, s) = \frac{\partial}{\partial\theta}\ell(\theta \,|\, s) = \frac{2s}{\theta} - \frac{4s}{1-\theta}$.

(b) Solve (with explanation) the corresponding Score Equation.

Solution. *The Score Equation is* $S(\theta \,|\, s) = 0$, *which is equivalent to* $2s(1-\theta) - 4s(\theta) = 0$, *or* $\theta = 2s/6s = 1/3$.

(c) Determine (with explanation) the MLE, $\hat{\theta}$, for θ.

Solution. *Here the derivative* $S(\theta \,|\, s)$ *is well-defined throughout* Ω. *And, the second derivative* $(\frac{\partial}{\partial\theta})^2\ell(\theta \,|\, s) = -2s\theta^{-2} - 4s(1-\theta)^{-2} < 0$ *for all* $\theta \in \Omega$ *and* $s \in S$. *And, on the boundary as* $\theta \to 0$ *or* $\theta \to 1$, *the likelihood goes to 0. Hence, the solution to the Score Equation must be a global maximum, so* $\hat{\theta} = 1/3$.

3. Let $\Omega = (0, \infty)$, $S = [6, \infty)$, and $P_\theta = \text{Uniform}[6,\, 5\theta + 6]$ for $\theta \in \Omega$. Suppose we observe the observations $x_1, x_2, \ldots, x_n$, with $x_i \geq 6$ for all i.

(a) Compute (with full explanation) the MLE, $\hat{\theta}$, for θ.

Solution. *The density of P_θ is equal to $1/5\theta$ for $6 \leq x_i \leq 5\theta + 6$, otherwise 0. Hence, the likelihood function $L(\theta \mid x_1, \ldots, x_n)$ is equal to $(1/5\theta)^n$ provided that $6 \leq x_i \leq 5\theta + 6$ for all i, i.e. $\max_{1 \leq i \leq n} x_i \leq 5\theta + 6$, otherwise it equals 0. Hence, the likelihood is maximised when $(1/5\theta)^n$ is as large as possible (i.e., θ is as small as possible), subject to the constraint that $\max_{1 \leq i \leq n} x_i \leq 5\theta + 6$, i.e. $5\theta + 6 \geq \max_{1 \leq i \leq n} x_i$, i.e. $\theta \geq \big[\big(\max_{1 \leq i \leq n} x_i\big) - 6\big]/5$. The smallest θ satisfying this constraint is $\hat{\theta} = \big[\big(\max_{1 \leq i \leq n} x_i\big) - 6\big]/5 = \frac{1}{5}\big[\max_{1 \leq i \leq n}(x_i - 6)\big]$, which is the MLE.*

(b) Compute (with explanation) the MLE for θ^2.

Solution. *Since the mapping $\theta \mapsto \theta^2$ is 1–1 on S, we can use the "Plug-In Estimator" as the MLE for θ^2. Thus the MLE for θ^2 is equal to $(\hat{\theta})^2 = \left(\frac{1}{5}\big[\max_{1 \leq i \leq n}(x_i - 6)\big]\right)^2 = \frac{1}{25}\big[\max_{1 \leq i \leq n}(x_i - 6)^2\big]$.*

4. Suppose we observe three observations: $x_1 = 2$, $x_2 = 3$, $x_3 = 7$.

(a) Compute $\overline{x}$ and S^2. [Provide actual numbers, not just formulae.]

Solution. $\overline{x} = \frac{1}{3}[2 + 3 + 7] = 12/3 = 4$.

$S^2 = \frac{1}{3-1}[(2-4)^2 + (3-4)^2 + (7-4)^2] = \frac{1}{2}[4 + 1 + 9] = 14/2 = 7$.

(b) Suppose the statistical model is a Location-Scale Model, with $\Omega = \mathbf{R} \times (0, \infty)$, and $P_{(\mu, \sigma^2)} = N(\mu, \sigma^2)$ for $(\mu, \sigma^2) \in \Omega$. Compute (with explanation) a 95% confidence interval for μ. [You should provide an explicit numerical formula, but you do not need to simplify arithmetic expressions. You may use the facts that if $T_2 \sim t(2)$, $T_3 \sim t(3)$, and $T_4 \sim t(4)$, then $P[T_2 \leq -2.92] \doteq P[T_3 \leq -2.35] \doteq P[T_4 \leq -2.13] \doteq 0.05$, and $P[T_2 \leq -4.30] \doteq P[T_3 \leq -3.18] \doteq P[T_4 \leq -2.78] \doteq 0.025$.]

Solution. *We know that under P_θ, $T \equiv \sqrt{n/S^2}\,(\overline{X} - \mu) \sim t(n-1)$, i.e. $T \equiv \sqrt{3/7}\,(\overline{X} - \mu) \sim t(2)$. Hence, $P[-4.30 < T < +4.30] = 1 - P[T \leq -4.30] - P[T \geq +4.30] = 1 - 2\,P[T \leq -4.30] \doteq 1 - 2(0.025) = 0.95$. Thus, $0.95 = P[-4.30 < \sqrt{3/7}\,(\overline{X} - \mu) < +4.30] = P[\overline{X} - 4.30\sqrt{7/3} < \mu < \overline{X} - 4.30\sqrt{7/3}]$. Hence, a 95% C.I. is $(\overline{x} - 4.30\sqrt{7/3},\ \overline{x} + 4.30\sqrt{7/3}) = (4 - 4.30\sqrt{7/3},\ 4 + 4.30\sqrt{7/3})$. [This equals $(-2.57,\ 10.57)$, but you don't need to compute that.]*

(c) Suppose the statistical model is a Location Model, with $\Omega = \mathbf{R}$, and $P_\theta = N(\theta, 4)$ for

$\theta \in \Omega$. Compute (with explanation) a P-value for the null hypothesis $H_0 : \theta = 6$ versus the alternative hypothesis $H_1 : \theta \neq 6$. [You may leave your answer in terms of the Φ function.]

Solution. *We know that under* P_6, $Z \equiv \sqrt{n/\sigma^2}\,(\overline{X} - 6) = \sqrt{3/4}\,(\overline{X} - 6) \sim N(0, 1)$. *The observed value of* Z *was* $\sqrt{3/4}\,(4 - 6) = -\sqrt{3}$. *The probability (under* P_6*) of observing a value which is at least as surprising, is equal to* $P[|Z| \geq \sqrt{3}] = 2\,\Phi(-\sqrt{3})$. *[This equals* 0.0833, *but you don't need to compute that.]*

5. Let $\Omega = S = \mathbf{R}$, with $P_\theta = \text{Uniform}[\theta - 3,\ \theta + 3]$ for $\theta \in \Omega$. Suppose we observe $x_1, x_2, \ldots, x_{100}$, and that $\overline{x} = 11$.

(a) Find $C_1 > 0$ and C_2 (which may depend on θ, but may not depend on $x_1, \ldots, x_{100}$) such that if $Z = C_1(\overline{X} - C_2)$, then under P_θ, Z has mean 0 and variance 1. [Here $\overline{X}$ stands for the corresponding random variable, as opposed to the observed value $\overline{x}$. Also, recall that the Uniform$[a, b]$ distribution has mean $(a + b)/2$, and variance $(b - a)^2/12$.]

Solution. *Here* P_θ *has mean* $[(\theta - 3) + (\theta + 3)]/2 = \theta$, *and variance* $[(\theta + 3) - (\theta - 3)]^2/12 = 6^2/12 = 36/12 = 3$. *Hence,* $\overline{X}$ *has mean* θ *and variance* $3/n = 3/100$. *Hence, if* $C_1 = 1/\sqrt{3/100} = 10/\sqrt{3}$ *and* $C_2 = \theta$, *then* $Z = C_1(\overline{X} - C_2) = 10(\overline{X} - \theta)/\sqrt{3}$ *has mean 0 and variance 1 under* P_θ.

(b) Compute (with explanation) an approximate 95% confidence interval for θ. [Hint: Use the C.L.T.]

Solution. *Since* $n = 100$ *is reasonably large, we can use the C.L.T. approximation to conclude that under* P_θ, $Z \approx N(0, 1)$, *i.e.* $10(\overline{X} - \theta)/\sqrt{3} \approx N(0, 1)$. *Thus* $0.95 \doteq P[-1.96 < (\overline{X} - \theta)(10/\sqrt{3}) < +1.96] = P[\overline{X} - (\sqrt{3}/10)\,1.96 < \theta < \overline{X} + (\sqrt{3}/10)\,1.96]$. *Hence, a 95% C.I. is* $(\overline{x} - 1.96\sqrt{3}/10,\ \overline{x} + 1.96\sqrt{3}/10) = (11 - 0.196\sqrt{3},\ 11 + 0.196\sqrt{3})$. *[This equals* $(10.66,\ 11.34)$, *but you don't need to compute that.]*

6. Suppose $\Omega = S = \mathbf{R}$, and we observe two observations x_1 and x_2, and the likelihood function is given by $L(\theta \,|\, x_1, x_2) = \exp[(x_1 - \theta)^2]\,\exp[2\theta x_2]$. Let $T(x_1, x_2) = x_1 - x_2$.

(a) Is T a sufficient statistic for θ? (Explain your reasoning.)

Solution. *Yes,* T *is sufficient. Indeed,* $L(\theta \,|\, x_1, x_2) = \exp[(x_1 - \theta)^2 + 2\theta x_2] = \exp[x_1^2 - 2\theta x_1 + \theta^2 + 2\theta x_2] = \exp[x_1^2 + \theta^2 - 2\theta T(x_1, x_2)] = h(x_1, x_2)\, g_\theta(T(x_1, x_2))$, *where* $h(x_1, x_2) = \exp[x_1^2]$, *and* $g_\theta(t) = \exp[\theta^2 - 2\theta t]$. *Hence, by the Factorisation Theorem,* T *is sufficient.*

(b) Is T a <u>minimal</u> sufficient statistic for θ? (Explain your reasoning.)

Solution. *Yes,* T *is minimal.*

Proof #1: Indeed, if $L(\theta \,|\, x_1, x_2) = K\,L(\theta \,|\, y_1, y_2)$ *for all* $\theta \in \Omega$, *then*

$$L(1 \,|\, x_1, x_2)/L(1 \,|\, y_1, y_2) = L(0 \,|\, x_1, x_2)/L(0 \,|\, y_1, y_2)\,.$$

Hence,

$$L(1 \,|\, x_1, x_2)/L(0 \,|\, x_1, x_2) = L(1 \,|\, y_1, y_2)/L(0 \,|\, y_1, y_2)\,,$$

i.e.

$$\exp[x_1^2 + 1^2 - 2(1)T(x_1, x_2)]/\exp[x_1^2 + 0^2 - 2(0)T(x_1, x_2)]$$

$$= \exp[y_1^2 + 1^2 - 2(1)T(y_1, y_2)]/\exp[y_1^2 + 0^2 - 2(0)T(y_1, y_2)]\,,$$

i.e. $\exp[1 - 2\,T(x_1, x_2)] = \exp[1 - 2\,T(y_1, y_2)]$. *It follows that* $1 - 2T(x_1, x_2) = 1 - 2T(y_1, y_2)$, *and so* $T(x_1, x_2) = T(y_1, y_2)$. *Hence,* T *is minimal.*

Proof #2: If $L(\theta \,|\, x_1, x_2) = K\,L(\theta \,|\, y_1, y_2)$ *for all* $\theta \in \Omega$, *then* $S(\theta \,|\, x_1, x_2) = S(\theta \,|\, y_1, y_2)$, *i.e.* $2\theta - 2\theta T(x_1, x_2) = 2\theta - 2\theta T(y_1, y_2)$, *and so* $T(x_1, x_2) = T(y_1, y_2)$. *Hence,* T *is minimal.*

Proof #3: The solution to the Score Equation is $\theta = T(x_1, x_2)$. *Hence, since equivalent likelihoods have the same Score Equation, they also have the same value of* T. *Hence,* T *is minimal.*