STA447/2006 HW#1 Solutions – OUTLINE ONLY

NOTE: What follows is just an outline of the homework solutions, taken mostly from the textbook's solutions manual. These solutions may be incorrect or incomplete; more complete explanations are required to earn full points on the homework.

9.4. (a) to go from 1 to 4 in three steps we must go 1,2,3,4 so $p^3(1,4) = (.4)^3 = .064$. (b) to go from 1 to 0 in three steps we may go 1,2,1,0 or 1,0,0,0 so $p^3(1,0) = (.4)(.6)^2 + .6 = .744$

9.8. (a) $\{1,5\}$ and $\{2,5\}$ are finite irreducible closed sets so all of these states are recurrent. $3 \to 1$ but $1 \not\to 3$ so 3 is transient.
(b)(a) $\{1,4\}$ and $\{2,5\}$ are finite irreducible closed sets so all of these states are recurrent. $3 \to 2$ but $2 \not\to 3$ so 3 is transient. $6 \to 1$ but $1 \not\to 6$ so 6 is transient.

Question 3: (a) States 1,6,7 are recurrent; states 2,3,4,5 are transient. (b) [Can use e.g. the expansion that $f_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}$.] Answers are $f_{11} = f_{21} = f_{31} = 1$, $f_{61} = f_{71} = 0$, $f_{51} = 1/3$, and $f_{41} = (1/2) + (1/2)(1/3) = 2/3$.

9.1. No. We first argue this intuitively: when $X_n = 1$ the last two results may be $0,1$ or $1,0$. In the first case we may jump only to 2 or 1, while in the second case we may only jump to 0 or 1. Thus it is not enough to know just current state. To get a formal contradiction we note that $X_1 = 2$, $X_2 = 1$ implies that $Y_0 = Y_1 = 1$, $Y_2 = 0$ so

$$P(X_3 = 2 | X_2 = 1, X_1 = 2) = 0 < P(X_3 = 2 | X_2 = 1)$$

9.36. (a) To increase the number of black balls by 1 we have to pick a white from the first and a black from the second. Using similar reasoning on the other cases we have

$$p(i, i+1) = \frac{(m-i)\cdot(b-i)}{m \cdot m} \quad p(i, i-1) = \frac{i \cdot (m-b+i)}{m \cdot m}$$
$$p(i,i) = \frac{i(b-i) + (m-i)(m-b+i)}{m \cdot m}$$

(b) Detailed balance is equivalent to $\pi(i+1)/\pi(i) = p(i,i+1)/p(i+1,i)$. To check this condition we compute

$$\begin{aligned}\frac{\pi(i+1)}{\pi(i)} &= \frac{b!}{(i+1)!(b-i-1)!}\frac{(2m-b)!}{(m-i-1)!(m-b+i+1)!} \\ &\quad \cdot \frac{i!(b-i)!}{b!}\frac{(m-i)!(m-b+i)!}{(2m-b)!} \\ &= \frac{i!(b-i)!(m-i)!(m-b+i)!}{(i+1)!(b-i-1)!(m-i-1)!(m-b+i+1)!} \\ &= \frac{(b-i)(m-i)}{(i+1)(m-b+i+1)} = \frac{p(i,i+1)}{p(i+1,i)}\end{aligned}$$

(c) If we pick m balls at random from the $2m$ to put in one urn then $\pi(i)$ gives the probability we will get exactly i black balls. If we put the balls in randomly and then switch two we still have a random arrangement so we have constructed a stationary distribution.

9.32. (a) $E_x X_1 = x + (N-x)/N - x/N = 1 + (1-2x)/N$. Taking x to be random with distribution equal to that of X_n the result follows. (b) The formula holds when $n = 0$. If it is true for n then

$$\begin{aligned}\mu_{n+1} &= 1 + (1-2/N)\left(\frac{N}{2} + (1-2/N)^n (x - N/2)\right) \\ &= \frac{N}{2} + \left(1 - \frac{2}{N}\right)^{n+1} (x - N/2)\end{aligned}$$

9.31. (a) When $u, v > 0$ the chain is irreducible so the conclusion follows from (4.7) and (4.5). (b) By the formula for the mean of the binomial $E_x X_1 = N\rho_x = vN + (1-u-v)x$. Iterating we have

$$E_x X_2 = vN + (1-u-v)vN + (1-u-v)^2 x$$
$$E_x X_3 = vN + (1-u-v)vN + (1-u-v)^2 vN + (1-u-v)^3 x$$

so $\lim_{n\to\infty} E_x X_n = vN \sum_{m=0}^{\infty} (1-u-v)^m = Nv/(u+v)$. A simple way to see this is to note that in the absence of mutation an individual in generation n s the same as a randomly chosen one from generation $n-1$, so its type is dictated by the first mutation encountered as we work backwards and that will be to a 1 with probability $v/(u+v)$.

9.35. Let j be a possible arrangement of the deck. There are exactly 52 arrangements k that can arise from j in one step. In the other direction there are 52 arrangements i that can lead to j in one step. Since each transition has probability 1/52, it follows that $\sum_i p(i,j) = 52 \cdot (1/52) = 1$. The matrix is doubly stochastic, so by a result in class the stationary distribution is uniform.

(You also have to prove the chain is irreducible and aperiodic, though that is not too hard.)