STA447/2006 (Stochastic Processes), Winter 2008

Homework #2

Due: In class by 6:10 p.m. **sharp** on Thursday February 14. (If you prefer, you may bring your assignment to the instructor's office, Sidney Smith Hall room 6024, any time before it is due; slide it under the door if he is not in.) **Warning: Late homeworks, even by one minute, will be penalised!** (See the "Grade-Related Course Policies".)

Reminder: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Copying other solutions is strictly prohibited!

THE ASSIGNMENT: [Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. Consider the variant of simple random walk given by $S = \mathbf{Z}$ (the set of all integers), with $p_{i,i-1} = p_{i,i} = p_{i,i+1} = 1/3$ for all $i \in S$. Suppose, hypothetically, that this chain had a stationary distribution π. Define β by $\beta_i = \pi_{i+1}$ for all $i \in S$.

(a) [5 points] Prove that β would also be a stationary distribution.

(b) [5 points] Prove that $\forall\ i, j \in \mathbf{Z}$, $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$, and $\lim_{n\to\infty} p_{ij}^{(n)} = \beta_j$.

(c) [5 points] Show that this leads to a contradiction (so actually this chain does not have a stationary distribution).

2. *Gibbs Sampler.* Let $S = \mathbf{Z} \times \mathbf{Z}$, and let $\pi : S \to (0,1)$ with $\sum_{(x,y)\in S} \pi(x,y) = 1$. Let Q be an (infinite) matrix of Markov chain transition probabilities, defined by

$$q_{(x,y),(x',y')} = \begin{cases} \dfrac{\pi(x',y)}{\sum_{z\in\mathbf{Z}} \pi(z,y)}, & y = y' \\ 0, & \text{otherwise} \end{cases}$$

Similarly, define R by

$$r_{(x,y),(x',y')} = \begin{cases} \dfrac{\pi(x,y')}{\sum_{z\in\mathbf{Z}} \pi(x,z)}, & x = x' \\ 0, & \text{otherwise} \end{cases}$$

Let P be the Markov chain corresponding to doing first Q and then R, i.e. $p_{(x,y),(x',y')} = \sum_{(z,w)\in S} q_{(x,y),(z,w)} r_{(z,w),(x',y')}$, or equivalently $P = QR$.

(a) [5 points] Prove that π is a stationary distribution for Q, and also for R.

(b) [5 points] Prove that π is a stationary distribution for P.

(c) [5 points] Prove that P is irreducible.

(d) [5 points] Prove that P is aperiodic.

(e) [5 points] Conclude that $\lim_{n\to\infty} p^{(n)}_{(x,y),(x',y')} = \pi_{(x',y')}$ for all $x, y, x', y' \in \mathbf{Z}$.

3. [5 points] Let $C \in \mathbf{R}$, and let $\{Z_i\}$ be an i.i.d. collection of random variables with $\mathbf{P}[Z_i = -1] = 3/4$ and $\mathbf{P}[Z_i = C] = 1/4$. Let $X_0 = 5$, and $X_n = 5 + Z_1 + Z_2 + \ldots + Z_n$ for $n \geq 1$. Find (with explanation) a value of C such that $\{X_n\}$ is a martingale.

4. [10 points] Text exercise 5.3 (p. 121).

5. Let $\{X_n\}$ be simple symmetric random walk, so $S = \mathbf{Z}$ and $p_{i,i-1} = p_{i,i+1} = 1/2$, and assume that $X_0 = a = 1$. Let $\sigma = \min\{n \geq 1 : X_n = 0 \text{ or } 3\}$, and let $\rho = \sigma - 1$.

(a) [5 points] Compute $\mathbf{E}(X_\sigma)$.

(b) [5 points] Compute $\mathbf{E}(X_\rho)$. [Hint: if $X_\sigma = 0$, then what does X_ρ equal?]

(c) [5 points] Does $\mathbf{E}(X_\sigma) = a$? Does $\mathbf{E}(X_\rho) = a$? Relate these facts to theorems from class about martingales and stopping times.

6. [10 points] Let $\{X_n\}$ be a non-negative submartingale, with $\mathbf{E}(X_n^2) < \infty$ for all n. Let $Y_n = (X_n)^2$. Prove that $\{Y_n\}$ is also a submartingale.