NOTE: What follows is just an outline of the homework solutions, taken from the textbook’s solutions manual. These solutions may be incorrect or incomplete; more complete explanations may be required to earn full points on the homework.

8.17. \( X_t \) is a birth and death chain. The death rates (jumping into the lake) are \( \mu_i = i \), while the birth rates (jumping out of the lake) are \( \lambda_i = 2(3 - i) \). Setting \( \pi(0) = c \) and plugging into the recursion (3.5) gives

\[
\begin{align*}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{6}{1} \cdot c = 6c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{4}{2} \cdot 6c = 12c \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{2}{3} \cdot 12c = 8c
\end{align*}
\]

Adding up the \( \pi \)'s gives \( (8 + 12 + 6 + 1) = 27c \) so \( c = 1/27 \) and we have

\[
\begin{align*}
\pi(3) &= \frac{8}{27} \\
\pi(2) &= \frac{12}{27} \\
\pi(1) &= \frac{6}{27} \\
\pi(0) &= \frac{1}{27}
\end{align*}
\]

(b) Each frog is a two state Markov chain that stays in the sun 2/3’s of the time and in the lake 1/3 of the time. Thus the number in the sun should be Binomial(3,2/3). Since the Binomial probabilities are

\[
\begin{align*}
\pi(3) &= (2/3)^3 \\
\pi(2) &= 3(2/3)^2(1/3) \\
\pi(1) &= 3(1/3)^2(2/3) \\
\pi(0) &= (1/3)^3
\end{align*}
\]

this agrees with the previous answer.
8.19. Let $X_t$ be the number of working machines. $X_t$ is a birth and death chain. Taking into account the number of repairmen working $\lambda_2 = 1/2$, $\lambda_1 = \lambda_0 = 1$. The death rate is proportional to the number of machines working so $\mu_1 = 1/20$, $\mu_2 = 2/10$ and $\mu_3 = 3/20$. Setting $\pi(0) = c$ and plugging into the recursion (3.5) gives

$$
\begin{align*}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{1}{1/20} \cdot c = 20c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{1}{2/20} \cdot 20c = 200c \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{1/2}{3/20} \cdot 200c = 2000c/3
\end{align*}
$$

Adding up the $\pi$’s gives $(2000 + 600 + 60 + 3)c/3 = 2663c/3$ so $c = 3/2663$ and we have

$$
\begin{align*}
\pi(3) &= \frac{2000}{2663} \\
\pi(2) &= \frac{600}{2663} \\
\pi(1) &= \frac{60}{2663} \\
\pi(0) &= \frac{3}{2663}
\end{align*}
$$

(b) $\pi(0) + \pi(1) = 63/2663 = .0237$ of the time. (c) $(6000 + 1200 + 60)/2663 = 7260/2663 = 2.726$.

8.25. Let $X_t$ be the number of customers in the system. $X_t$ is a birth and death chain with $\lambda_n = \lambda$ for all $n \geq 0$, and $\mu_n = \mu + (n-1)\delta$. It follows from (3.6) that

$$
\pi(n + 1) = \frac{\lambda_n}{\mu_{n+1}} \cdot \pi(n)
$$

If $\delta > 0$, we have $\lambda_n/\mu_{n+1} \to 0$ as $n \to \infty$. Hence if $N$ is large enough and $n \geq N$ then $\lambda_n/\mu_{n+1} \leq 1/2$ and the desired conclusion follows from the argument in Example 3.5. (b) When $\delta = \mu$, $\mu_n = n\mu$ and

$$
\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \frac{\lambda^n}{\mu^n} \cdot \frac{1}{1 \cdot 2 \cdots n} = \frac{(\lambda/\mu)^n}{n!}
$$

It follows that the stationary distribution is Poisson with mean $\lambda/\mu$. 

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5.4. Using (1.1) we have

\[ P(M > n) = P(t_1 + \cdots + t_n \leq 1) = \int_{t_1=0}^{1} \int_{t_2=0}^{1-t_1} \cdots \int_{t_n=0}^{1-(t_1+\cdots+t_{n-1})} 1 \, dt_n \cdots \, dt_2 \, dt_1 = \frac{1}{n!} \]

From this and (1.3) it follows that

\[ EM = \sum_{n=0}^{\infty} P(M > n) = \sum_{n=0}^{\infty} \frac{1}{n!} = e \]

(b) Wald's equation, (1.5), implies that

\[ ET_M = EM \cdot Et_i \]

Since \( t_i \) is uniform on \((0,1)\) we have \( Et_i = 1/2 \). Using (a) now

\[ E(T_M - 1) = \left( e \cdot \frac{1}{2} \right) - 1 = .35914 \]

6.1. Using the definition of conditional probability gives \( P(B_s = x | B_t = z) \) is

\[
\frac{\exp(-x^2/2s)}{(2\pi s)^{1/2}} \cdot \frac{\exp(-(z-x)^2/2(t-s))}{(2\pi (t-s))^{1/2}} \cdot \frac{(2\pi t)^{1/2}}{\exp(-z^2/2(t-s))}
\]

\[ = (2\pi)^{-1/2} \left( \frac{t}{s(t-s)} \right)^{1/2} \exp \left\{ -\frac{x^2}{2s} - \frac{(x-z)^2}{2(t-s)} - \frac{z^2}{2(t-s)} \right\} \]

The expression in braces is

\[-\frac{1}{2} \left( \frac{t}{s(t-s)} \right) x^2 - \frac{2}{(t-s)} xy + \frac{s}{t(t-s)} z^2 \]

\[ = -\frac{t}{2s(t-s)} \left( x - \frac{zs}{t} \right)^2 \]

so the distribution is normal\((zs/t, s(t-s)/t)\).
6.4. (a) \( EY_t = \int_0^t EB_s \, ds = 0 \). (b) To do this, we use a trick

\[
EY_t^2 = E \left( \int_0^t B_s \, ds \right)^2 = E \left( \int_0^t B_r \, dr \right) \left( \int_0^t B_s \, ds \right) = E \left( \int_0^t \int_0^t B_r B_s \, dr \, ds \right) = 2 \int_0^t \int_0^t EB_r B_s \, dr \, ds = 2 \int_0^t \int_0^t r \, dr \, ds = \int_0^t s^2 \, ds = t^3 / 3
\]

(c) Clearly \( EY_t^2 = EY_s^2 + EY_s(Y_t - Y_s) \), so we only have to compute the second term. To do this we imitate the computation in (b)

\[
EY_s(Y_t - Y_s) = E \left( \int_0^s B_r \, dr \cdot \int_s^t B_u \, du \right) = \int_0^s \int_t^s EB_r B_u \, du \, dr = \int_0^s \int_s^t dr \, ds = (t - s) \int_0^s r \, dr = (t - s)s^2 / 2
\]

6.39. By (4.3) we want to find a probability distribution so that the two stocks are each martingales, i.e.,

\[
20p_1 + 10p_2 - 16p_3 = 0 \quad -20p_1 + 5p_2 + 10p_3 = 0
\]

Substituting \( p_3 = 1 - p_1 - p_2 \) we have

\[
36p_1 + 26p_2 = 16 \quad -30p_1 - 5p_2 = -10
\]

Multiplying the second equation by 6/5 and adding it to the first we have

\[
20p_2 = 4 \quad \text{so} \quad p_2 = .2 \]

Solving now gives \( p_1 = .3 \) and \( p_3 = .5 \). An option to buy Netscape at 50 pays off 0 in case 1, 5 in case 2, and 10 in case 3, so the option is worth \( 0p_1 + 5p_2 + 10p_3 = 0 + 1 + 5 = 6 \).