STA 447/2006S, Winter 2008: In-Class Test
(DRAFT) SOLUTIONS

1. [8 points] Let (p_{ij}) be the transition probabilities for random walk on the graph whose vertices are $V = \{1, 2, 3, 4\}$, with a single edge between each of the four pairs (1,2), (2,3), (3,1), and (3,4), and no other edges. Compute (with full explanation) $\lim_{n\to\infty} p_{13}^{(n)}$.

> **Solution:** *Since $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$, the graph is connected, so the random walk is irreducible. Also, state 3 is aperiodic since e.g. it is possible to get from 3 to 3 in two steps ($3 \to 4 \to 3$) or in three steps ($3 \to 1 \to 2 \to 3$), and $\gcd(2, 3) = 1$. So, by irreducibility, all states are aperiodic. Also, from class, the chain has stationary distribution given by $\pi_u = d(u)/\sum_v d(v) = d(u)/2|E|$. In this case, $d(1) = d(2) = 2$, $d(3) = 3$, and $d(4) = 1$, so $\sum_v d(v) = 2 + 2 + 3 + 1 = 8$ (or equivalently, $2|E| = 2 \cdot 4 = 8$). Hence, by the Markov chain convergence theorem, $\lim_{n\to\infty} p_{13}^{(n)} = \pi_3 = d(3)/8 = 3/8$.*

2. Consider the Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities given by $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

(a) [4 points] Compute (with explanation) f_{12} (i.e., the probability, starting from 1, that the chain will eventually visit 2).

> **Solution:** *When the chain leaves the state 1, it goes to either 2 or 3 and then stays there. So, $f_{12} = \mathbf{P}(X_1 = 2 \,|\, X_0 = 1, X_1 \neq 1) = p_{12}/(1 - p_{11}) = 1/3/(1 - 1/6) = (1/3)/(5/6) = 2/5$.*
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> *Alternatively, since $f_{32} = 0$ (because $p_{33} = 1$), we have that $f_{12} = p_{12} + \sum_{j\neq 2} p_{1j} f_{j2} = p_{12} + p_{11} f_{12} + p_{13} f_{32} = (1/3) + (1/6) f_{12} + (1/2)(0)$, so $(5/6) f_{12} = 1/3$, so $f_{12} = 2/5$.*

(b) [3 points] Prove that $p_{12}^{(n)} \geq 1/3$, for any positive integer n.

> **Solution:** *Since $p_{22} = 1$, it follows that $p_{22}^{(m)} = 1$ for all $m \geq 0$. Then by the Chapman-Kolmogorov equations, $p_{12}^{(n)} \geq p_{12}\, p_{22}^{(n-1)} = (1/3)(1) = 1/3$.*

(c) [2 points] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

> **Solution:** *Since $p_{12}^{(n)} \geq 1/3$, we have $\sum_{n=1}^{\infty} p_{12}^{(n)} \geq \sum_{n=1}^{\infty} (1/3) = \infty$, so $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$.*

(d) [3 points] Relate the answers in parts (a) and (c) to theorems from class about when $f_{ij} = 1$ and when $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.

Solution: *We proved in class that $f_{ij} = 1$ if and only if $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$, provided that either $i = j$, or the chain is irreducible. In this case, we have $f_{12} < 1$ but $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$. However, this is not a contradiction since $1 \neq 2$, and also the chain is not irreducible since e.g. $f_{32} = 0$.*

3. Let $S = \mathbf{Z}$ (the set of all integers), and let $h : S \to [0,1]$ with $\sum_{i \in S} h(i) = 1$. Consider the transition probabilities on S given by $p_{ij} = (1/4)\min(1,\, h(j)/h(i))$ if $j = i-2, i-1, i+1$, or $i+2$, and $p_{ii} = 1 - p_{i,i-2} - p_{i,i-1} - p_{i,i+1} - p_{i,i+2}$, and $p_{ij} = 0$ whenever $|j - i| \geq 3$.

(a) [10 points] Assuming that $h(i) > 0$ for all i, prove that $\lim_{n\to\infty} p_{ij}^{(n)} = h(j)$ for all $i, j \in S$. (Carefully justify each step.)

Solution: *(i) The chain is irreducible since $p_{i,i+1} > 0$ and $p_{i,i-1} > 0$ for all i, so if $j > i$ then $p_{ij}^{(j-i)} \geq p_{i,i+1}p_{i+1,i+2}\ldots p_{j-1,j} > 0$, while if $j < i$ then $p_{ij}^{(j-i)} \geq p_{i,i-1}p_{i-1,i-2}\ldots p_{j+1,j} > 0$. (Also $p_{ii}^{(2)} \geq p_{i,i+1}p_{i+1,i} > 0$.)*

(ii) The chain is aperiodic since $\sum_{i\in S} h(i) = 1$ implies $\lim_{j\to\infty} h(j) = 0$, which means there must be some $i \in S$ with $h(i+1) < h(i)$, whence $p_{i,i+1} < 1/4$, whence $p_{ii} > 0$, so the period of state i is 1, and then by irreducibility the period of every state is 1.

[Or, alternatively, the period of state i is 1 since $p_{ii}^{(2)} \geq p_{i,i+1}p_{i+1,i} > 0$ and $p_{ii}^{(3)} \geq p_{i,i+1}p_{i+1,i+2}p_{i+2,i} > 0$, and $gcd(2,3) = 1$.]

(iii) With $\pi_i = h(i)$, the chain satisfies detailed balance $\pi_i p_{ij} = \pi_j p_{ij}$ for all $i, j \in S$. Indeed, the statement is trivial if $i = j$, and both sides are 0 if $|j - i| > 2$. For $1 \leq |j - i| \leq 2$, we have that $\pi_i p_{ij} = h(i)(1/4)\min(1,\, h(j)/h(i)) = (1/4)\min(h(i),\, h(j))$ which is symmetric in i and j, so $\pi_i p_{ij} = \pi_j p_{ij}$.

(iv) Since the chain satisfies detailed balance with $\pi_i = h(i)$, therefore π is a stationary distribution.

(v) Hence, by the Markov chain convergence theorem, $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j = h(j)$ for all $i, j \in S$.

(b) [5 points] Show by example that part (a) might be false if we do not assume that $h(i) > 0$ for all i. [For definiteness, we take $\min(1,\, h(j)/h(i)) \equiv 1$ whenever $h(i) = 0$.]

Solution: *Suppose that, say, $h(5) = h(6) = 0$, but $h(3), h(4), h(7) > 0$. Then $p_{35} = p_{45} = p_{46} = 0$. Furthermore we always have $p_{ij} = 0$ for $j > i + 2$. It follows that $p_{ij} = 0$ whenever $i \leq 4$ and $j \geq 5$. Hence, $f_{ij} = 0$ whenever $i \leq 4$ and $j \geq 5$. In particular, $p_{47}^{(n)} = 0$ for all n, so $\lim_{n\to\infty} p_{47}^{(n)} = 0 \neq h(7)$ since $h(7) > 0$.*

4. Consider a Markov chain $\{X_n\}$ with state space $S = \{1, 2, 3, 4, 5\}$, $X_0 = 4$, and transition probabilities specified by $p_{11} = p_{55} = 1$, $p_{21} = 5/7$, $p_{24} = p_{25} = 1/7$, $p_{31} = p_{32} = p_{33} = p_{34} = p_{35} = 1/5$, and $p_{43} = p_{45} = 1/2$. Let $T = \min\{n \geq 1 : X_n = 1 \text{ or } 5\}$.

(a) [8 points] Determine (with full explanation) whether or not $\{X_n\}$ is a martingale.

> **Solution:** *Yes, $\{X_n\}$ is a martingale. Since $\{X_n\}$ is a Markov chain, and clearly $\mathbf{E}|X_n| \leq 5 < \infty$, it suffices to show that $\sum_{j \in S} j\, p_{ij} = i$ for all $i \in S$.*
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> *$i = 1$: $\sum_{j \in S} j\, p_{ij} = 1\, p_{11} = 1(1) = 1 = i$.*
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> *$i = 2$: $\sum_{j \in S} j\, p_{ij} = 1\, p_{21} + 4\, p_{24} + 5\, p_{25} = 1(5/7) + 4(1/7) + 5(1/7) = 14/7 = 2 = i$.*
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> *$i = 3$: $\sum_{j \in S} j\, p_{ij} = 1\, p_{31} + 2\, p_{32} + 3\, p_{33} + 4\, p_{34} + 5\, p_{35} = 1(1/5) + 2(1/5) + 3(1/5) + 4(1/5) + 5(1/5) = 15/5 = 3 = i$.*
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> *$i = 4$: $\sum_{j \in S} j\, p_{ij} = 3\, p_{43} + 5\, p_{45} = 3(1/2) + 5(1/2) = 8/2 = 4 = i$.*
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> *$i = 5$: $\sum_{j \in S} j\, p_{ij} = 5\, p_{55} = 5(1) = 5 = i$.*
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> *So, $\sum_{j \in S} j\, p_{ij} = i$ for all $i \in S$, so $\{X_n\}$ is a martingale.*

(b) [4 points] Compute $\mathbf{P}(X_T = 5)$. [Hint: part (a) might help.]

> **Solution:** *Let $q = \mathbf{P}(X_T = 5)$. Then $\mathbf{P}(X_T = 1) = 1 - q$. Hence, $\mathbf{E}(X_T) = 5q + 1(1-q)$. But $\{X_n\}$ is a bounded martingale, and T is a stopping time (since $\{T = n\}$ depends only on $X_0, \ldots, X_n$), and $\mathbf{P}(T < \infty) = 1$ (since e.g. $\mathbf{P}(T = n+1 \,|\, T > n,\ X_n = i) \geq 1/7$ for $i = 2, 3, 4$, so $\mathbf{P}(T \geq n) \leq (1 - 1/7)^n$, so $\mathbf{P}(T = \infty) = 0$). Hence, we must have $\mathbf{E}(X_T) = \mathbf{E}(X_0) = 4$, so $5q + 1(1-q) = 4$, i.e. $4q + 1 = 4$, so $4q = 3$, so $q = 3/4$.*

5. Consider a Markov chain $\{X_n\}$ on the state space $S = \{0, 1, 2, 3, \ldots\}$, with $X_0 = 100$, and $p_{ij} = 1/(2i+1)$ if $0 \leq j \leq 2i$, otherwise $p_{ij} = 0$.

(a) [5 points] Prove that $\{X_n\}$ is a martingale. (You may assume without proof that $\mathbf{E}|X_n| < \infty$ for all n.)

> **Solution:** *We compute that $\sum_j j\, p_{ij} = \sum_{j=0}^{2i} j/(2i+1) = (2i)(2i+1)/2/(2i+1) = i$ for any $i \in S$, so $\{X_n\}$ is a martingale.*

(b) [5 points] Prove that $\mathbf{P}(\exists n \geq 1 : X_n = 1000) < 1/6$. [Hint: the martingale maximal inequality might help.]

> **Solution:** *Since $\{X_n\}$ is a non-negative martingale, we have by the martingale maximal inequality that $\mathbf{P}(\exists n \geq 1 : X_n = 1000) \leq \mathbf{P}(\max_n X_n \geq 1000) \leq$*

$\mathbf{E}(X_0)/1000 = 100/1000 = 1/10 < 1/6$.

6. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$.

(a) [6 points] Compute the conditional probability $q_\lambda \equiv \mathbf{P}(N(4) = 1 \,|\, N(5) = 3)$.

Solution: $q_\lambda = \mathbf{P}(N(4) = 1 \,|\, N(5) = 3) = \frac{\mathbf{P}(N(4)=1,\ N(5)=3)}{\mathbf{P}(N(5)=3)} = \frac{\mathbf{P}(N(4)=1,\ N(5)-N(4)=2)}{\mathbf{P}(N(5)=3)} = \frac{(e^{-4\lambda}(4\lambda)^1/1!)(e^{-\lambda}\lambda^2/2!)}{e^{-5\lambda}(5\lambda)^3/3!} = \frac{(4)^1/1!)(1/2!)}{(5)^3/3!} = \frac{4/2}{125/6} = 24/250 = 12/125$.

(b) [2 points] Compute $q_{2\lambda} \,/\, q_\lambda$. (That is, determine the fraction by which the probability in part (a) changes if we replace λ by 2λ.)

Solution: *By part (a),* $\frac{q_{2\lambda}}{q_\lambda} = \frac{12/125}{12/125} = 1$, *i.e. the probability does not change if we replace* λ *by* 2λ *(or by any other value).*