CIRCLE ONE: **STA447** **STA2006** Student #: ______________________

Surname: ______________________ Given name(s): ______________________

STA 447/2006S, Winter 2008: In-Class Test

(February 28, 2008, 6:10 p.m. Time: 130 minutes.)

(Questions: 6; Pages: 6; Total points: 65.)

NO AIDS ALLOWED – NOT EVEN CALCULATORS.

1. [8 points] Let (p_{ij}) be the transition probabilities for random walk on the graph whose vertices are $V = \{1, 2, 3, 4\}$, with a single edge between each of the four pairs (1,2), (2,3), (3,1), and (3,4), and no other edges. Compute (with full explanation) $\lim_{n\to\infty} p_{13}^{(n)}$.

2. Consider the Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities given by $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

(a) [4 points] Compute (with explanation) f_{12} (i.e., the probability, starting from 1, that the chain will eventually visit 2).

(b) [3 points] Prove that $p_{12}^{(n)} \geq 1/3$, for any positive integer n.

(c) [2 points] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

(d) [3 points] Relate the answers in parts (a) and (c) to theorems from class about when $f_{ij} = 1$ and when $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.

3. Let $S = \mathbf{Z}$ (the set of all integers), and let $h : S \to [0,1]$ with $\sum_{i \in S} h(i) = 1$. Consider the transition probabilities on S given by $p_{ij} = (1/4)\min(1,\ h(j)/h(i))$ if $j = i-2, i-1, i+1$, or $i+2$, and $p_{ii} = 1 - p_{i,i-2} - p_{i,i-1} - p_{i,i+1} - p_{i,i+2}$, and $p_{ij} = 0$ whenever $|j-i| \geq 3$.

(a) [10 points] Assuming that $h(i) > 0$ for all i, prove that $\lim_{n\to\infty} p_{ij}^{(n)} = h(j)$ for all $i, j \in S$. (Carefully justify each step.)

(b) [5 points] Show by example that part (a) might be false if we do not assume that $h(i) > 0$ for all i. [For definiteness, we take $\min(1,\ h(j)/h(i)) \equiv 1$ whenever $h(i) = 0$.]

4. Consider a Markov chain $\{X_n\}$ with state space $S = \{1, 2, 3, 4, 5\}$, $X_0 = 4$, and transition probabilities specified by $p_{11} = p_{55} = 1$, $p_{21} = 5/7$, $p_{24} = p_{25} = 1/7$, $p_{31} = p_{32} = p_{33} = p_{34} = p_{35} = 1/5$, and $p_{43} = p_{45} = 1/2$. Let $T = \min\{n \geq 1 : X_n = 1 \text{ or } 5\}$.

(a) [8 points] Determine (with full explanation) whether or not $\{X_n\}$ is a martingale.

(b) [4 points] Compute $\mathbf{P}(X_T = 5)$. [Hint: part (a) might help.]

5. Consider a Markov chain $\{X_n\}$ on the state space $S = \{0, 1, 2, 3, \ldots\}$, with $X_0 = 100$, and $p_{ij} = 1/(2i + 1)$ if $0 \leq j \leq 2i$, otherwise $p_{ij} = 0$.

(a) [5 points] Prove that $\{X_n\}$ is a martingale. (You may assume without proof that $\mathbf{E}|X_n| < \infty$ for all n.)

(b) [5 points] Prove that $\mathbf{P}(\exists n \geq 1 : X_n = 1000) < 1/6$. [Hint: the martingale maximal inequality might help.]

6. Let $\{N(t)\}_{t \geq 0}$ be a Poisson process with rate $\lambda > 0$.

(a) [6 points] Compute the conditional probability $q_\lambda \equiv \mathbf{P}(N(4) = 1 \,|\, N(5) = 3)$.

(b) [2 points] Compute $q_{2\lambda} \,/\, q_\lambda$. (That is, determine the fraction by which the probability in part (a) changes if we replace λ by 2λ.)

[END]