1. [8 points] Let \( (p_{ij}) \) be the transition probabilities for random walk on the graph whose vertices are \( V = \{1, 2, 3, 4\} \), with a single edge between each of the four pairs \( (1,2), (2,3), (3,1), \) and \( (3,4) \), and no other edges. Compute (with full explanation) \( \lim_{n \to \infty} p_{13}^{(n)} \).
2. Consider the Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities given by $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

(a) [4 points] Compute (with explanation) $f_{12}$ (i.e., the probability, starting from 1, that the chain will eventually visit 2).

(b) [3 points] Prove that $p_{12}^{(n)} \geq 1/3$, for any positive integer $n$.

(c) [2 points] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

(d) [3 points] Relate the answers in parts (a) and (c) to theorems from class about when $f_{ij} = 1$ and when $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$. 

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3. Let $S = \mathbb{Z}$ (the set of all integers), and let $h : S \to [0, 1]$ with $\sum_{i \in S} h(i) = 1$. Consider the transition probabilities on $S$ given by $p_{ij} = (1/4) \min(1, h(j)/h(i))$ if $j = i-2, i-1, i+1$, or $i+2$, and $p_{ii} = 1 - p_{i-2,i} - p_{i-1,i} - p_{i+1,i} - p_{i+2,i}$, and $p_{ij} = 0$ whenever $|j - i| \geq 3$.

(a) [10 points] Assuming that $h(i) > 0$ for all $i$, prove that $\lim_{n \to \infty} p_{ij}^{(n)} = h(j)$ for all $i, j \in S$. (Carefully justify each step.)

(b) [5 points] Show by example that part (a) might be false if we do not assume that $h(i) > 0$ for all $i$. [For definiteness, we take $\min(1, h(j)/h(i)) \equiv 1$ whenever $h(i) = 0$.]
4. Consider a Markov chain \( \{X_n\} \) with state space \( S = \{1, 2, 3, 4, 5\} \), \( X_0 = 4 \), and transition probabilities specified by 
\[ p_{11} = p_{55} = 1, \quad p_{21} = 5/7, \quad p_{24} = p_{25} = 1/7, \quad p_{31} = p_{32} = p_{33} = p_{34} = p_{35} = 1/5, \quad \text{and} \quad p_{43} = p_{45} = 1/2. \]
Let \( T = \min\{n \geq 1 : X_n = 1 \text{ or } 5\} \).

(a) [8 points] Determine (with full explanation) whether or not \( \{X_n\} \) is a martingale.

(b) [4 points] Compute \( P(X_T = 5) \). [Hint: part (a) might help.]
5. Consider a Markov chain \( \{X_n\} \) on the state space \( S = \{0, 1, 2, 3, \ldots\} \), with \( X_0 = 100 \), and \( p_{ij} = 1/(2i + 1) \) if \( 0 \leq j \leq 2i \), otherwise \( p_{ij} = 0 \).

(a) [5 points] Prove that \( \{X_n\} \) is a martingale. (You may assume without proof that \( \mathbb{E}|X_n| < \infty \) for all \( n \).)

(b) [5 points] Prove that \( \mathbb{P}(\exists n \geq 1 : X_n = 1000) < 1/6 \). [Hint: the martingale maximal inequality might help.]
6. Let \( \{N(t)\}_{t \geq 0} \) be a Poisson process with rate \( \lambda > 0 \).

(a) [6 points] Compute the conditional probability \( q_\lambda \equiv P(N(4) = 1 \mid N(5) = 3) \).

(b) [2 points] Compute \( q_{2\lambda} / q_\lambda \). (That is, determine the fraction by which the probability in part (a) changes if we replace \( \lambda \) by \( 2\lambda \).)