STA 3431S, Winter 2009: In-Class Test
(DRAFT) SOLUTIONS

1. [5 points] Consider a linear congruential pseudorandom number generator, with parameters $m = 100$, $a = 11$, and $b = 3$. Suppose the initial seed value is $x_0 = 1$. Compute the first two values U_1 and U_2 that the generator will produce.

> **Solution:** *Here* $x_1 = ax_0 + b = 11(1) + 3 = 14$. *And,* $x_2 = ax_1 + b = 11(14) + 3 = 157 \equiv 57 \pmod{m}$. *So,* $U_1 = x_1/m = 0.14$, *and* $U_2 = x_2/m = 0.57$.

2. [5 points] Let $g(y) = e^y(1 + \sin(y))\mathbf{1}_{[0,8]}(y)$, let $\pi(y) = c\,g(y)$ be a probability density (with unknown normalising constant c), and let $h(y) = y^2$. Suppose we simulate two samples $x_1, x_2 \sim$ Exponential(3), and observe that $x_1 = 2$ and $x_2 = 6$. Compute the Importance Sampler estimate of $\mathbf{E}_\pi(h)$ based on these two samples.

> **Solution:** *Here the* x_i *samples are from the density* $f(x) = 3e^{-3x}$ ($x > 0$), *so the Importance Sampler estimate is*

$$\frac{\sum_{i=1}^{2} h(x_i)\, g(x_i)/f(x_i)}{\sum_{i=1}^{2} g(x_i)/f(x_i)} \;=\; \frac{\big(h(2)g(2)/f(2)\big) + \big(h(6)g(6)/f(6)\big)}{\big(g(2)/f(2)\big) + \big(g(6)/f(6)\big)}$$

$$= \frac{\big(2^2e^2(1+\sin(2)) \;/\; 3e^{-3(2)}\big) + \big(6^2e^6(1+\sin(6)) \;/\; 3e^{-3(6)}\big)}{\big(e^2(1+\sin(2)) \;/\; 3e^{-3(2)}\big) + \big(e^6(1+\sin(6)) \;/\; 3e^{-3(6)}\big)}$$

$$= \frac{4e^8(1+\sin(2)) + 36e^{24}(1+\sin(6))}{e^8(1+\sin(2)) + e^{24}(1+\sin(6))}$$

$$= \frac{4(1+\sin(2)) + 36e^{16}(1+\sin(6))}{(1+\sin(2)) + e^{16}(1+\sin(6))}\,.$$

3. Consider the statistical model that $Y_i \sim$ Exponential(a) for $i = 1, 2, 3$ (conditionally independently), where $a \in \mathbf{R}$ is unknown. Let a have prior density given by $f(a) \propto (1 + a^2)\mathbf{1}_{[1,4]}(a)$. Suppose we then observe the data values $Y_1 = 2$, $Y_2 = 3.5$, $Y_3 = 6.5$.

(a) [5 points] Conditional on this data, specify (up to a normalising constant) the posterior density of a.

> **Solution:** *Here the likelihood function is* $\prod_{i=1}^{3} ae^{-aY_i}$, *so since "posterior equals prior times likelihood", the posterior density is*

$$\pi(a) \;=\; c\,f(a)\prod_{i=1}^{3} ae^{-aY_i} \;=\; c\,(1+a^2)\mathbf{1}_{[1,4]}(a)\,a^3e^{-a(2+3.5+6.5)} \;=\; c\,(1+a^2)a^3e^{-12\,a}\mathbf{1}_{[1,4]}(a)\,.$$

(b) [10 points] Write a complete R program to run an Independence Sampler algorithm with proposal distribution $N(0,1)$, to estimate the posterior mean of $\log(a)$. (Your program should output the final estimate, but you don't need to bother with standard errors or confidence intervals or acceptance rates or plots or any other "extras".)

Solution:

```
M = 1100    # run length
B = 100     # burn-in
g = function(a) {
  if ( (a<1) || (a>4) )
    return(0)
  else
    return( (1+a∧2) * a∧3 * exp(-12*a) )
}
alist = rep(0,M)
a = runif(1,1,4)    # overdispersed starting distribution: Uniform[1,4]
for (i in 1:M) {
  y = rnorm(1)    # proposal
  A = (g(y) * dnorm(a)) / (g(a) * dnorm(y))
  U = runif(1)
  if (U<A)
    a = y    # accept proposal
  alist[i] = a
}
print(mean(log(a[(B+1):M])))    # output the estimate
```

4. [5 points] Suppose from a Markov chain run with $M = 200$ and $B = 100$, we obtain the estimates that $\mathbf{E}_\pi(h) \approx 20$, and $\mathrm{Var}_\pi(h) \approx 9$, and $\mathrm{Corr}_\pi[h(X_n), h(X_{n+1})] \approx 0.6$, and $\mathrm{Corr}_\pi[h(X_n), h(X_{n+2})] \approx 0.3$, and $\mathrm{Corr}_\pi[h(X_n), h(X_{n+k})] \approx 0$ for all $k \geq 3$. In terms of this, provide an approximate 95% confidence interval for $\mathbf{E}_\pi(h)$, clearly stating any assumptions that you require.

Solution: *Here we estimate* $v = \frac{1}{M-B}(\mathrm{Var}_\pi(h))(\mathit{varfact}) \approx \frac{1}{100}\,9\Big(1 + 2\sum_{k=1}^{\infty} \mathrm{Corr}_\pi[h(X_n), h(X_{n+k})]\Big) \approx 0.09\big(1 + 2(0.6 + 0.3 + 0)\big) = 0.09(2.8) = 0.252.$ *Then, we know from class that if a CLT holds (e.g. the chain is geometrically ergodic, and* $\mathbf{E}_\pi(|h|^{2+\delta}) < \infty$ *for some* $\delta > 0$*), and if the burn-in* B *and effective run length* $M - B$ *are both sufficiently large, then an approximate 95% confidence interval for* $\mathbf{E}_\pi(h)$ *is given by:*

$$(e - 1.96\sqrt{v},\ e + 1.96\sqrt{v}) \;=\; (20 - 1.96\sqrt{0.252},\ 20 + 1.96\sqrt{0.252}) \;\doteq\; (19.016,\ 20.984)\,.$$

5. [5 points] Consider a usual Parallel Tempering (MCMCMC) algorithm, with five temperatures $\tau = 1, 2, 3, 4, 5$, and with $\pi_\tau(x) = c_\tau \left(e^{-|x|^3}\right)^{1/\tau}$ for $x \in \mathbf{R}$, for some $c_\tau > 0$. Assume that, as usual, the temperatures for an attempted swap are chosen uniformly at random from $\{1, 2, 3, 4, 5\}$. Suppose that at time n, the algorithm is in the state $X_n = (1, 2, 2, 1, 4)$, and then proposes to swap the values corresponding to $\tau = 3$ and $\tau = 5$. Compute the probability that this proposed swap will be accepted.

Solution: *We know from class that for symmetrically-chosen temperature pairs* τ *and* τ'*, the acceptance probability for a proposed swap is*

$$\min\Big(1,\ \frac{\pi_\tau(X_{n,\tau'})\,\pi_{\tau'}(X_{n,\tau})}{\pi_\tau(X_{n,\tau})\,\pi_{\tau'}(X_{n,\tau'})}\Big) \;=\; \min\Big(1,\ \frac{\pi_3(4)\,\pi_5(2)}{\pi_3(2)\,\pi_5(4)}\Big)$$

$$=\; \min\Big(1,\ \frac{c_3(\pi(4))^{1/3}\,c_5(\pi(2))^{1/5}}{c_3(\pi(2))^{1/3}\,c_5(\pi(4))^{1/5}}\Big) \;=\; \min\Big(1,\ \frac{(e^{-4^3})^{1/3}\,(e^{-2^3})^{1/5}}{(e^{-2^3})^{1/3}\,(e^{-4^3})^{1/5}}\Big)$$

$$=\; \min\big(1,\ e^{-(4^3/3)-(2^3/5)+(2^3/3)+(4^3/5)}\big) \;=\; \min\big(1,\ e^{-(64/3)-(8/5)+(8/3)+(64/5)}\big)$$

$$=\; \min\big(1,\ e^{-(64-8)((1/3)-(1/5))}\big) \;=\; \min\big(1,\ e^{-56(2/15)}\big)$$

$$=\; \min\big(1,\ e^{-112/15}\big) \;=\; e^{-112/15} \;\doteq\; 0.0005718\,.$$

6. Let $\mathcal{X} = [0, 1]$. Define Markov chain transition probabilities $P(x, \cdot)$ as follows. If $x = 1/m$ for some positive integer m, then $P(x, \cdot) = x^2\,\text{Uniform}[0, 1] + (1 - x^2)\,\delta_{1/(m+1)}(\cdot)$, i.e. it is a mixture distribution (where $\delta_{1/(m+1)}$ is the point-mass [degenerate] distribution at the point $1/(m + 1)$). For all other $x \in \mathcal{X}$, $P(x, \cdot) = \text{Uniform}[0, 1]$.

(a) [5 points] Does this chain have a stationarity distribution Π, and if so what is it?

Solution: *Yes,* $\Pi = \text{Uniform}[0, 1]$ *is stationary, since if we choose* $X_0 \sim \text{Uniform}[0, 1]$*, then* $\mathbf{P}(X_0 = 1/m$ *for some positive integer* $m) = 0$*, so*

$$\mathbf{P}(X_1 \in S) \;=\; \int_{x\in\mathcal{X}} P(x, S)\,\Pi(dx) \;=\; \int_{x\in\mathcal{X}} (\text{Uniform}[0, 1])(S)\,\Pi(dx)$$

$$=\; (\text{Uniform}[0, 1])(S) \;=\; \Pi(S)$$

as required.

(b) [5 points] Determine (with explanation) the set $G \subseteq \mathcal{X}$ of states $x \in \mathcal{X}$, such that $\lim_{n\to\infty} P^n(x, S) = \Pi(S)$ for all (measurable) $S \subseteq \mathcal{X}$ if and only if $x \in G$.

Solution: *If* $x \notin \{1/2, 1/3, 1/4, \ldots\}$*, then* $P^n(x, \cdot) = \Pi(\cdot)$ *for all* $n \geq 1$*, so indeed* $\lim_{n\to\infty} P^n(x, S) = \Pi(S)$ *for all (measurable)* $S \subseteq \mathcal{X}$.

However, if $x = 1/m$ *for some integer* $m \geq 2$*, then* $\mathbf{P}(X_n = 1/(m+n)$ *for all* $n \geq 1 \mid X_0 = x) = \prod_{n=0}^{\infty}(1-(m+n)^{-2}) \geq \prod_{j=2}^{\infty}(1-j^{-2}) = \exp\left(\sum_{j=2}^{\infty}\log(1-j^{-2})\right) \geq \exp\left(-\sum_{j=2}^{\infty}(j^{-2}+\frac{1}{2}j^{-4})\right) \equiv e^{-K} > 0$*, where* $K = \sum_{j=2}^{\infty}(j^{-2}+\frac{1}{2}j^{-4}) < \infty$*. (We have used that for* $|y| < 1$, $\log(1+y) = y - (y^2/2) + (y^3/3) - (y^4/4) + \ldots \geq y - (y^2/2)$*. Alternatively, simply note that* $\prod_{j=2}^{\infty}(1-j^{-2}) > 0$ *since* $(1-j^{-2}) > 0$ *for all* $2 \leq j < \infty$*, and* $\sum_{j=2}^{\infty} j^{-2} < \infty$*.) In particular, if* $S = \{1/2, 1/3, \ldots\}$*, then* $P^n(x, S) \geq P^n(x, \{1/(m+n)\}) \geq e^{-K} \neq 0 = \Pi(S)$*, so we do not have* $\lim_{n\to\infty} P^n(x, S) = \Pi(S)$ *for all (measurable)* $S \subseteq \mathcal{X}$*.*

So, $G = \mathcal{X} \setminus \{1/2, 1/3, 1/4, \ldots\} = \{x \in \mathcal{X} : x \neq 1/m$ *for any integer* $m \geq 2\}$*.*