1. Let $\Omega = \{1, 2, 3\}$. Let $F = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$, which you may assume is a $\sigma$-algebra. Let $P : F \to [0, 1]$ by $P(\emptyset) = 0$, $P(\{1\}) = 4/5$, $P(\{2, 3\}) = 1/5$, and $P(\{1, 2, 3\}) = 1$. Let $X, Y : \Omega \to R$ by $X(1) = 5$, $X(2) = 10$, $X(3) = 10$, $Y(1) = 2$, $Y(2) = 4$, $Y(3) = 6$.

(a) [3 points] Verify that $P$ is countably additive on $F$.

**Solution.** If $A_1, A_2, \ldots$ are disjoint, then either (i) at most one of the $A_i$ is non-empty, say $A_1$, in which case additivity is trivial since $P(\bigcup_n A_n) = P(A_1) = \sum_n P(A_n)$, or (ii) precisely two of the $A_i$ are non-empty, with one of the non-empty $A_i$ being $\{1\}$ and the other being $\{2, 3\}$, in which case $P(\bigcup_n A_n) = P(\{1\} \cup \{2, 3\}) = P(\{1, 2, 3\}) = 1 = 4/5 + 1/5 = P(\{1\}) + P(\{2, 3\}) = \sum_n P(A_n)$. So, in either case, $P(\bigcup_n A_n) = \sum_n P(A_n)$, i.e. $P$ is countably additive.

(b) [3 points] Is $X$ a valid random variable on $(\Omega, F, P)$?

**Solution.** Yes, since $\{X \leq a\}$ can only be $\emptyset$ (if $x < 5$) or $\{1\}$ (if $5 \leq x < 10$) or $\{1, 2, 3\}$ (if $x \geq 10$), all of which are in $F$.

(c) [3 points] Is $Y$ a valid random variable on $(\Omega, F, P)$?

**Solution.** No, since e.g. $\{Y \leq 5\} = \{1, 2\}$ which is not in $F$.

(d) [1 point] Compute $P(X > 8)$.

**Solution.** $P(X > 8) = P(\omega \in \Omega : X(\omega) > 8) = P(\{2, 3\}) = 1/5$.

2. [5 points] Let $(\Omega, F, P)$ be as in Question 1. Let $G$ be the collection of all subsets of $\Omega$ (so, $F \subseteq G$). Determine (with explanation) which one of the following statements is true (recalling that “extension from $F$ to $G$” means a countably additive probability measure on $G$, which agrees with the original $P$ when restricted to $F$):

(i) $P$ has no possible extension from $F$ to $G$,

or (ii) $P$ has one unique extension from $F$ to $G$,

or (iii) $P$ has more than one possible extension from $F$ to $G$.

**Solution.** (iii) is true. For example, let $P_1$ be defined by $P_1(\{1\}) = 4/5$, $P_1(\{2\}) = 0$, $P_1(\{3\}) = 1/5$, and additivity, and let $P_2$ be defined by $P_2(\{1\}) = 4/5$, $P_2(\{2\}) = 1/5$, $P_2(\{3\}) = 0$, and additivity. Then $P_1$ and $P_2$ are both countably additive probability measures (by construction). Also $P_1(\{1\}) = P_2(\{1\}) = 4/5$, and $P_1(\{2, 3\}) = P_2(\{2, 3\}) = 1/5$, so $P_1$ and $P_2$ both agree with $P$ on $F$. Thus, $P_1$ and $P_2$ are two different extensions of $P$ from $F$ to $G$, so there is more than one possible extension. [NOTE: the uniqueness part of the Extension Theorem does NOT apply, since $G \not\subseteq \sigma(F)$.]

3. [5 points] Let $(\Omega, F, P)$ be any valid probability triple for which $\Omega = \{1, 2, 3, \ldots\}$, and $F$ is the collection of all subsets of $\Omega$. For each $n \in N$, let $A_n = \{n, n + 1, n + 2, \ldots\}$. Is it necessarily true that $\lim_{n \to \infty} P(A_n) = 0$? Why or why not?

**Solution.** Yes, the statement is true. Here $A_{n+1} \subseteq A_n$, and $\bigcap_n A_n = \emptyset$. Hence, by continuity of probabilities, $\lim_{n \to \infty} P(A_n) = P(\bigcap_n A_n) = P(\emptyset) = 0$. 

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4. Let $(\Omega, \mathcal{F}, P)$ be the probability triple defined by $\Omega = \{1, 2, 3, 4\}$, and $\mathcal{F}$ is the collection of all subsets of $\Omega$, and $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = 1/4$. Let $A_n = \{1\}$ for $n$ odd, and $A_n = \{2, 3\}$ for $n$ even.

(a) [4 points] Are $A_1, A_2, A_3, \ldots$ independent?

Solution. No, e.g. $P(A_1 \cap A_2) = P(\{1\} \cap \{2, 3\}) = P(\emptyset) = 0$, but $P(A_1)P(A_2) = P(\{1\})P(\{2, 3\}) = (1/4)(1/4 + 1/4) = 1/8 \neq 0$.

(b) [4 points] Compute $P\left(\lim\inf_{n \to \infty} A_n\right)$.

Solution. Since $1 \notin A_n$ for all even $n$, and $2, 3 \notin A_n$ for all odd $n$, therefore $\{A_n \ a.a.\}$ is empty, i.e. $\lim\inf_{n} A_n = \emptyset$, so $P(\lim\inf_{n} A_n) = P(\emptyset) = 0$.

(c) [2 points] Compute $\lim\inf_{n \to \infty} P(A_n)$.

Solution. Here $P(A_n) = 1/4$ for $n$ odd, and $P(A_n) = 1/4 + 1/4 = 1/2$ for $n$ even. So, $P(A_n)$ oscillates between $1/4$ and $1/2$. Hence, $\lim\inf_{n} P(A_n) = 1/4$.

(d) [2 points] Compute $\lim\sup_{n \to \infty} P(A_n)$.

Solution. As above, $P(A_n)$ oscillates between $1/4$ and $1/2$. Hence, $\lim\sup_{n} P(A_n) = 1/2$.

(e) [4 points] Compute $P\left(\lim\sup_{n \to \infty} A_n\right)$.

Solution. Since $1 \in A_n$ for all odd $n$, and $2, 3 \in A_n$ for all even $n$, therefore $\{A_n \ i.o.\} = \{1, 2, 3\}$, so $P(\lim\sup_{n} A_n) = P(\{1, 2, 3\}) = 1/4 + 1/4 + 1/4 = 3/4$. [Note: since $\{A_n\}$ are not independent, we cannot use the Borel-Cantelli Lemma.]
