STA2111F, Fall 2011, Mid-Term Test: SOLUTIONS.

1. Let $\Omega = \{1,2,3\}$. Let $\mathcal{F} = \Big\{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\Big\}$, which you may assume is a σ-algebra. Let $\mathbf{P} : \mathcal{F} \to [0,1]$ by $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}(\{1\}) = 4/5$, $\mathbf{P}(\{2,3\}) = 1/5$, and $\mathbf{P}(\{1,2,3\}) = 1$. Let $X, Y : \Omega \to \mathbf{R}$ by $X(1) = 5$, $X(2) = 10$, $X(3) = 10$, $Y(1) = 2$, $Y(2) = 4$, $Y(3) = 6$.

(a) [3 points] Verify that $\mathbf{P}$ is countably additive on $\mathcal{F}$.

> **Solution.** *If $A_1, A_2, \ldots$ are disjoint, then either (i) at most one of the A_i is non-empty, say A_1, in which case additivity is trivial since $\mathbf{P}\left(\bigcup_n A_n\right) = \mathbf{P}(A_1) = \sum_n \mathbf{P}(A_n)$, or (ii) precisely two of the A_i are non-empty, with one of the non-empty A_i being $\{1\}$ and the other being $\{2,3\}$, in which case $\mathbf{P}\left(\bigcup_n A_n\right) = P(\{1\} \cup \{2,3\}) = P\{1,2,3\}) = 1 = 4/5 + 1/5 = P(\{1\}) + P(\{2,3\}) = \sum_n \mathbf{P}(A_n)$. So, in either case, $\mathbf{P}\left(\bigcup_n A_n\right) = \sum_n \mathbf{P}(A_n)$, i.e. $\mathbf{P}$ is countable additive.*

(b) [3 points] Is X a valid random variable on $(\Omega, \mathcal{F}, \mathbf{P})$?

> **Solution.** *Yes, since $\{X \le x\}$ can only be $\emptyset$ (if $x < 5$) or $\{1\}$ (if $5 \le x < 10$) or $\{1,2,3\}$ (if $x \ge 10$), all of which are in $\mathcal{F}$.*

(c) [3 points] Is Y a valid random variable on $(\Omega, \mathcal{F}, \mathbf{P})$?

> **Solution.** *No, since e.g. $\{Y \le 5\} = \{1,2\}$ which is not in $\mathcal{F}$.*

(d) [1 point] Compute $\mathbf{P}(X > 8)$.

> **Solution.** $\mathbf{P}(X > 8) = \mathbf{P}(\omega \in \Omega : X(\omega) > 8\} = \mathbf{P}(\{2,3\}) = 1/5$.

2. [5 points] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be as in Question 1. Let $\mathcal{G}$ be the collection of <u>all</u> subsets of Ω (so, $\mathcal{F} \subseteq \mathcal{G}$). Determine (with explanation) which <u>one</u> of the following statements is true (recalling that "extension from $\mathcal{F}$ to $\mathcal{G}$" means a countably additive probability measure on $\mathcal{G}$, which agrees with the original $\mathbf{P}$ when restricted to $\mathcal{F}$):
(i) $\mathbf{P}$ has no possible extension from $\mathcal{F}$ to $\mathcal{G}$,
or (ii) $\mathbf{P}$ has one unique extension from $\mathcal{F}$ to $\mathcal{G}$,
or (iii) $\mathbf{P}$ has more than one possible extension from $\mathcal{F}$ to $\mathcal{G}$.

> **Solution.** *(iii) is true. For example, let $\mathbf{P}_1$ be defined by $\mathbf{P}_1\{1\} = 4/5$, $\mathbf{P}_1\{2\} = 0$, $\mathbf{P}_1\{3\} = 1/5$, and additivity, and let $\mathbf{P}_2$ be defined by $\mathbf{P}_2\{1\} = 4/5$, $\mathbf{P}_2\{2\} = 1/5$, $\mathbf{P}_2\{3\} = 0$, and additivity. Then $\mathbf{P}_1$ and $\mathbf{P}_2$ are both countably additive probability measures (by construction). Also $\mathbf{P}_1\{1\} = \mathbf{P}_2\{1\} = 4/5$, and $\mathbf{P}_1\{2,3\} = \mathbf{P}_2\{2,3\} = 1/5$, so $\mathbf{P}_1$ and $\mathbf{P}_2$ both agree with $\mathbf{P}$ on $\mathcal{F}$. Thus, $\mathbf{P}_1$ and $\mathbf{P}_2$ are two different extensions of $\mathbf{P}$ from $\mathcal{F}$ to $\mathcal{G}$, so there is more than one possible extension. [NOTE: the uniqueness part of the Extension Theorem does NOT apply, since $\mathcal{G} \not\subseteq \sigma(\mathcal{F})$.]*

3. [5 points] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be any valid probability triple for which $\Omega = \{1,2,3,\ldots\}$, and $\mathcal{F}$ is the collection of all subsets of Ω. For each $n \in \mathbf{N}$, let $A_n = \{n, n+1, n+2, \ldots\}$. Is it necessarily true that $\lim_{n\to\infty} \mathbf{P}(A_n) = 0$? Why or why not?

> **Solution.** *Yes, the statement is true. Here $A_{n+1} \subseteq A_n$, and $\bigcap_n A_n = \emptyset$. Hence, by continuity of probabilities, $\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\bigcap_n A_n) = \mathbf{P}(\emptyset) = 0$.*

4. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability triple defined by $\Omega = \{1, 2, 3, 4\}$, and $\mathcal{F}$ is the collection of all subsets of Ω, and $\mathbf{P}(\{1\}) = \mathbf{P}(\{2\}) = \mathbf{P}(\{3\}) = \mathbf{P}(\{4\}) = 1/4$. Let $A_n = \{1\}$ for n odd, and $A_n = \{2, 3\}$ for n even.

(a) [4 points] Are $A_1, A_2, A_3, \ldots$ independent?

Solution. *No, e.g.* $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(\{1\} \cap \{2, 3\}) = \mathbf{P}(\emptyset) = 0$, *but* $\mathbf{P}(A_1)\,\mathbf{P}(A_2) = \mathbf{P}(\{1\})\,\mathbf{P}(\{2, 3\}) = (1/4)(1/4 + 1/4) = 1/8 \neq 0$.

(b) [4 points] Compute $\mathbf{P}\left(\liminf_{n\to\infty} A_n\right)$.

Solution. *Since* $1 \notin A_n$ *for all even* n*, and* $2, 3 \notin A_n$ *for all odd* n*, therefore* $\{A_n \text{ a.a.}\}$ *is empty, i.e.* $\liminf_n A_n = \emptyset$*, so* $\mathbf{P}(\liminf_n A_n) = \mathbf{P}(\emptyset) = 0$.

(c) [2 points] Compute $\liminf_{n\to\infty} \mathbf{P}(A_n)$.

Solution. *Here* $\mathbf{P}(A_n) = 1/4$ *for* n *odd, and* $\mathbf{P}(A_n) = 1/4 + 1/4 = 1/2$ *for* n *even. So,* $\mathbf{P}(A_n)$ *oscillates between* $1/4$ *and* $1/2$*. Hence,* $\liminf_n \mathbf{P}(A_n) = 1/4$.

(d) [2 points] Compute $\limsup_{n\to\infty} \mathbf{P}(A_n)$.

Solution. *As above,* $\mathbf{P}(A_n)$ *oscillates between* $1/4$ *and* $1/2$*. Hence,* $\limsup_n \mathbf{P}(A_n) = 1/2$.

(e) [4 points] Compute $\mathbf{P}\left(\limsup_{n\to\infty} A_n\right)$.

Solution. *Since* $1 \in A_n$ *for all odd* n*, and* $2, 3 \in A_n$ *for all even* n*, therefore* $\{A_n \text{ i.o.}\} = \{1, 2, 3\}$*, so* $\mathbf{P}(\limsup_n A_n) = \mathbf{P}(\{1, 2, 3\}) = 1/4 + 1/4 + 1/4 = 3/4$*. [Note: since* $\{A_n\}$ *are* <u>not</u> *independent, we* <u>cannot</u> *use the Borel-Cantelli Lemma.]*