1. Let \( \{X_n\} \) be the usual Gambler’s ruin Markov chain with \( p = 1/2, a = 5, \) and \( c = 20. \) Let \( T = \inf\{n \geq 0 : X_n = 0 \text{ or } c\} \) as usual, and let \( U = T - 1.\)

(a) [2] Compute \( E(X_T).\)

(b) [5] Compute \( E(X_U).\) [Hint: If \( X_T = c, \) then what must \( X_U \) equal? Or, if \( X_T = 0, \) then what must \( X_U \) equal?]

(c) [3] Determine if \( E(X_T) = E(X_0), \) and if \( E(X_U) = E(X_0). \) Relate these facts to the Optional Stopping Corollary.

2. [8] Let \( \{X_n\} \) be simple symmetric random walk, with \( X_0 = 0, \) and let \( S = \inf\{n \geq 0 : X_n = -5\}. \) Prove that \( \lim_{M \to \infty} E(X_M | S > M) = \infty. \) [Hint: Let \( T_M = \min(S, M), \) and apply the Optional Stopping Lemma.]

3. Let \( \{X_n\} \) be a Markov chain on the state space \( S = \{1, 2, 3, \ldots, 100\}, \) with initial probabilities given by \( \nu_{30} = \nu_{40} = 1/2, \) and with transition probabilities given by \( p_{1,1} = p_{100,100} = 1, \) \( p_{99,100} = p_{99,98} = 1/2, \) and for \( 2 \leq i \leq 98, \) \( p_{i,i-1} = 2/3 \) and \( p_{i,i+2} = 1/3. \) Let \( T = \inf\{n \geq 0 : X_n = 1 \text{ or } 100\}. \)

(a) [4] Compute \( P(X_4 = 42).\)

(b) [6] Determine whether or not \( \{X_n\} \) is a martingale.

(c) [6] Compute \( P(X_T = 1). \) [Hint: Don’t forget the Optional Stopping Corollary.]

4. [8] Let \( \{B_t\}_{t \geq 0} \) be Brownian motion. Compute \( \text{Var}(B_5B_8), \) the variance of \( B_5B_8. \) [Hint: You may use without proof that if \( Z \sim \text{Normal}(0, 1), \) then \( E(Z) = E(Z^3) = 0, \) \( E(Z^2) = 1, \) and \( E(Z^4) = 3. \) And don’t forget that \( B_8 = B_5 + (B_8 - B_5). \)]
5. [8] Let \( \{B_t\}_{t \geq 0} \) be Brownian motion. Let \( \theta \in \mathbb{R} \), and let \( Z_t = \exp(\theta B_t - \theta^2 t/2) \). Prove that \( \{Z_t\}_{t \geq 0} \) is a martingale. [Hint: You may use without proof the fact that if \( W \sim N(\mu, \sigma^2) \), then \( \mathbb{E}[e^{aW}] = e^{\mu a + \sigma^2 a^2/2} < \infty \).]

6. Let \( \{B_t\}_{t \geq 0} \) be Brownian motion, let \( X_t = X_0 \exp(\mu t + \sigma B_t) \) be the stock price model (where \( \sigma > 0 \)), and let \( D_t = e^{-rt} X_t \) be the discounted stock price.

(a) [4] Show that if \( \mu = r - \sigma^2/2 \), then \( \{D_t\} \) is a martingale. [Hint: Don’t forget the previous question.]

(b) [10] Show that if \( \mu = r - \sigma^2/2 \), then

\[
\mathbb{E}\left[e^{-rS} \max(0, X_S - K)\right] = X_0 \Phi\left(\frac{(r + \sigma^2/2)S - \log(K/X_0)}{\sigma \sqrt{S}}\right) - e^{-rS} K \Phi\left(\frac{(r - \sigma^2/2)S - \log(K/X_0)}{\sigma \sqrt{S}}\right),
\]

where \( \Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv \) is the cdf of a standard normal distribution. [Hint: Write the expectation as an integral with respect to the density function for \( B_S \). Then, break up the integral into the part where \( X_S - K \geq 0 \) and the part where \( X_S - K < 0 \).] [Note: this is the famous “Black-Scholes formula”. You do not need to memorise it!]

7. Let \( \{N(t)\}_{t \geq 0} \) be a Poisson process with intensity \( \lambda = 3 \).

(a) [5] Compute \( \mathbb{P}[N(6) = 2 \mid N(8) = 4] \).

(b) [5] Compute \( \mathbb{P}[N(6) = 2 \mid N(8) = 4, N(3) = 1] \).

(c) [6] Compute \( \mathbb{E}\left([N(8) - N(5)][N(7) - N(2)]\right) \). [Hint: You may use without proof the fact that if \( Y \sim \text{Poisson}(m) \), then \( \mathbb{E}(Y) = \text{Var}(Y) = m \).]

REMINDER: The final exam will be on Thursday Apr 12, from 7:00 – 10:00 p.m., in room 200 of Brennan Hall, 81 St. Mary Street (2nd floor). Bring your student card.