STA447/2006 (Stochastic Processes), Winter 2016

Homework #2

Due: In class by 6:10 p.m. **sharp** on Thursday March 15. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. [4] Try out the Java applet at: www.probability.ca/met
Explain briefly (in a few sentences) what the applet illustrates, and what options you tried, and what you found. [Or, if you can't get Java applets to run on your computer, then instead you can: (a) Explain how you tried to get the applet to run, and (b) Explain in 4–5 complete sentences the basic idea of the Metropolis algorithm.]

2. [4] Try out the Java applet at: www.probability.ca/gamone
Explain briefly (in a few sentences) what the applet illustrates, and what options you tried, and what you found. [Or, if you can't get Java applets to run on your computer, then instead you can: (a) Explain how you tried to get the applet to run, and (b) Explain in 4–5 complete sentences the basic idea of the Gambler's Ruin problem.]

3. Consider the undirected graph on the vertices $V = \{1, 2, 3, 4, 5\}$, with weights given by $w(1, 2) = w(2, 1) = w(2, 3) = w(3, 2) = w(1, 3) = w(3, 1) = w(3, 4) = w(4, 3) = w(3, 5) = w(5, 3) = 1$, and $w(u, v) = 0$ otherwise.

(a) [2] Draw a picture of this graph.

(b) [10] Compute (with full explanation) $\lim_{n\to\infty} \mathbf{P}[X_n = 3]$, where $\{X_n\}$ is the usual (simple) random walk on this graph.

4. [10] Let $\{X_n\}$ be simple random walk on $S = \mathbf{Z}$, with parameter $p = 2/3$, and with $X_0 = 5$. For any $i \in \mathbf{Z}$, let $T_i = \inf\{n \geq 1 : X_n = i\}$. Compute (with full explanation) $\mathbf{P}_5(T_0 < \infty)$. [Hint: First, use the Gambler's Ruin formula to compute $\mathbf{P}_5(T_c < T_0)$, and hence $\mathbf{P}_5(T_0 < T_c)$, for any $c > 6$. Then, consider (with justification) the limit as $c \to \infty$.]

5. [A special case of the *Gibbs Sampler.*] Let $S = \mathbf{Z} \times \mathbf{Z}$, and let $f : S \to (0, \infty)$ be some function from S to the positive real numbers. Let $K = \sum_{(x,y) \in S} f(x,y)$, and assume that $K < \infty$. For $x, y \in \mathbf{Z}$, let $C(x) = \sum_{w \in \mathbf{Z}} f(x,w)$, and $R(y) = \sum_{z \in \mathbf{Z}} f(z,y)$. Consider a Markov chain on S with transition probabilities given by

$$p_{(x,y),(z,w)} \;=\; \begin{cases} \dfrac{f(z,w)}{2\,C(x)} + \dfrac{f(z,w)}{2\,R(y)}\,, & x = z \text{ and } y = w \\ \dfrac{f(z,w)}{2\,C(x)}\,, & x = z \text{ and } y \neq w \\ \dfrac{f(z,w)}{2\,R(y)}\,, & x \neq z \text{ and } y = w \\ 0\,, & \text{otherwise} \end{cases}$$

(a) [5] Prove that $\sum\limits_{(z,w) \in S} p_{(x,y),(z,w)} = 1$ for all $(x,y) \in S$.

(b) [5] Show that the chain is reversible with respect to $\pi_{(x,y)} = \frac{f(x,y)}{K}$.

(c) [10] Compute $\lim\limits_{n \to \infty} p^{(n)}_{(x,y),(z,w)}$ for all $x, y, z, w \in \mathbf{Z}$ (carefully justifying each step).

6. Let $\{Z_i\}$ be an i.i.d. collection of random variables with $\mathbf{P}[Z_i = -1] = 3/4$ and $\mathbf{P}[Z_i = C] = 1/4$, for some $C > 0$. Let $X_0 = 5$, and $X_n = 5 + Z_1 + Z_2 + \ldots + Z_n$ for $n \geq 1$. Finally, let $T = \inf\{n \geq 1 : X_n = 0 \text{ or } Z_n > 0\}$.

(a) [5] Find (with explanation) a value of C such that $\{X_n\}$ is a martingale.

(b) [2] For this value of C, compute (with explanation) $\mathbf{E}(X_9)$.

(c) [3] For this value of C, compute (with explanation) $\mathbf{E}(X_T)$. [Hint: is T bounded?]

7. Consider simple random walk with $X_0 = 2$ and $p = 2/3$.

(a) [3] Compute $P(X_2 = i)$ for all $i \in \mathbf{Z}$.

(b) [3] Compute $E(X_2)$ using the explicit formula $\sum_i i\, P(X_2 = i)$, together with the probability values found in part (a).

(c) [4] Compute $E(X_2)$ using Wald's Theorem (with explanation), and see if you get the same answer as in part (b).

8. Consider a branching process with $X_0 = 2$, and with offspring distribution μ satisfying that $\mu\{0\} = 1/2$ and $\mu\{1\} = 1/3$ and $\mu\{2\} = 1/6$.

(a) [5] Compute $P(X_1 = i)$ for all $i \in \{0, 1, 2, \ldots\}$.

(b) [5] Compute $P(X_2 = i)$ for all $i \in \{0, 1, 2, \ldots\}$.