STA447/2006 Midterm, February 16, 2017

(2 hours; 5 questions; 6 pages; total points = 52)

LAST NAME: ____________ **GIVEN NAMES:** ____________ **STUDENT #:** ____________

Class (circle one): **STA447** **STA2006**

Do not open this booklet until told to do so. Answer all questions.

You may use results from class. Aids allowed: NONE.

Point values for each question are indicated [in square brackets].

You should explain all of your solutions clearly.

You may continue on the back of the page if necessary (write "OVER").

DO NOT WRITE BELOW THIS LINE.

Question	Score
1(a)	/3
1(b)	/2
1(c)	/2
1(d)	/3
1(e)	/3
1(f)	/2
1(g)	/2
2(a)	/3
2(b)	/3
2(c)	/3

Question	Score
3	/6
4	/5
5(a)	/3
5(b)	/3
5(c)	/3
5(d)	/3
5(e)	/3
TOTAL:	/52

1. Suppose there are 10 lily pads arranged in a circle, numbered consecutively clockwise from 1 to 10. A frog begins on lily pad #1. Each second, the frog jumps one pad clockwise with probability 1/4, or two pads clockwise with probability 3/4.

(a) [**3**] Specify a state space S, initial probabilities $\{\nu_i\}$, and transition probabilities $\{p_{ij}\}$, with respect to which this process is a Markov chain.

(b) [**2**] Determine if this Markov chain is irreducible.

(c) [**2**] Determine if this Markov chain is aperiodic, or if not then what its period equals.

(d) [**3**] Determine whether or not $\sum_{n=1}^{\infty} p_{15}^{(n)} = \infty$.

(e) [**3**] Either find a stationarity distribution $\{\pi_i\}$ for this chain, or prove that no stationary distribution exists.

(f) [**2**] Determine whether or not $\lim_{n\to\infty} p_{15}^{(n)}$ exists, and if so what it equals.

(g) [**2**] Determine whether or not $\lim_{n\to\infty} \frac{1}{2}[p_{15}^{(n)} + p_{15}^{(n+1)}]$ exists, and if so what it equals.

2. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) [**3**] There is a state $k \in S$ such that if the chain is started at k, then there is a positive probability that the chain will visit k exactly twice more (and then never again).

(b) [**3**] The chain is irreducible and transient, and there are $k, \ell \in S$ with $f_{k\ell} = 1$.

(c) [**3**] The chain is irreducible and transient, and is reversible with respect to some probability distribution π.

3. **[6]** Let $S = \{1, 2, 3, \ldots\}$, with $\pi_i = 2/3^i$ for all $i \in S$. Find (with proof) explicit transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$. [Hint: Don't forget the Metropolis (MCMC) algorithm.]

4. **[5]** Consider the undirected graph with vertex set $V = \{1, 2, 3, 4, 5\}$, and an undirected edge (of weight 1) between each of the following six pairs of vertices (and no other edges): (1,2), (2,3), (3,4), (4,5), (1,3), and (3,5). Let $\{p_{ij}\}_{i,j \in V}$ be the transition probabilities for random walk on this graph. Compute (with full explanation) $\lim_{n\to\infty} p_{23}^{(n)}$, or prove that this limit does not exist.

5. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, \ldots\}$ of all positive integers, which is also a martingale. Assume $X_0 = 5$, and that there is $c > 0$ such that $p_{i,i-1} = c$ and $p_{i,i+2} = 1 - c$ for all $i \geq 2$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } X_n \geq 10\}$.

(a) [**3**] Determine (with explanation) what c must equal. [Hint: remember that $\{X_n\}$ is a martingale.]

(b) [**3**] Determine (with explanation) what p_{11} must equal. [Hint: again, remember that $\{X_n\}$ is a martingale.]

(c) [**3**] Determine (with explanation) the value of $\mathbf{E}(X_3)$.

(d) [**3**] Determine (with explanation) the value of $\mathbf{E}(X_T)$.

(e) [**3**] Prove or disprove that $\sum_{n=1}^{\infty} p_{55}^{(n)} = \infty$.

[END OF EXAMINATION: total points = 52]