STA447/2006 (Stochastic Processes), Winter 2018

Homework #1

(5 questions; 2 pages; total points = 60)

Due: In class by 6:10 p.m. **sharp** on Thursday January 25. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying any other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. Consider a (discrete-time) Markov chain $\{X_n\}$ on the state space $S = \{1, 2, 3, 4\}$, with transition probabilities given by

$$(\, p_{ij} \,) \;=\; \begin{pmatrix} 1/3 & 1/6 & 1/2 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 2/5 & 3/5 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

(a) [**4**] Compute (with explanation) $p_{43}^{(3)} \equiv \mathbf{P}(X_3 = 3 \,|\, X_0 = 4)$. **Note:** You should compute this exactly, by hand; you should not do it numerically on a computer. (But you don't have to simplify the final fractions if you don't want to.)

(b) [**2**] Compute f_{21}. (Hint: perhaps set $C = \{1, 2, 3\}$.)

(c) (optional, zero points) Write a computer program to numerically estimate $p_{43}^{(3)}$ and $p_{43}^{(5)}$. How close is your estimate of $p_{43}^{(3)}$ to the exact answer from part (a)?

2. Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

(a) [**3**] Compute (with explanation) f_{12}.

(b) [**2**] Prove that $p_{12}^{(n)} \geq 1/3$, for all positive integers n.

(c) [**2**] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

(d) [**2**] Why do the answers in parts (c) and (a) not contradict the implication (1) $\Longrightarrow$ (5) in the Stronger Recurrence Theorem?

3. Suppose a fair six-sided die is repeatedly rolled, at times $0, 1, 2, 3, \ldots$. (So, each roll is independently equally likely to be 1, 2, 3, 4, 5, or 6.) Let X_n be the largest value that appears among all the rolls up to time n.

(a) **[6]** Find (with justification) a state space S, initial probabilities $\{\nu_i\}$, and transition probabilities $\{p_{ij}\}$, for which $\{X_n\}$ is Markov chain.

(b) **[4]** Compute the two different two-step transition probabilities $\{p_{35}^{(2)}\}$ and $\{p_{15}^{(2)}\}$.

(c) **[3]** Compute the single three-step transition probability $\{p_{15}^{(3)}\}$.

4. For each of the following sets of conditions, either provide (with explanation) an example of Markov chain transition probabilities $\{p_{ij}\}$ on some state space S such that the conditions are satisfied, or prove that no such Markov chain exists.

(a) **[3]** $3/4 < p_{12}^{(n)} < 1$ for all $n \geq 1$.

(b) **[3]** $p_{11} > 1/2$, and the state 1 is transient.

(c) **[3]** $f_{12} = 1/2$, and $f_{13} = 2/3$.

(d) **[3]** $p_{12}^{(n)} \geq 1/4$ and $p_{21}^{(n)} \geq 1/4$ for all $n \geq 1$, and the state 1 is transient.

(e) **[3]** S is finite, and $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$.

(f) **[3]** S is infinite, and $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$.

5. Consider a Markov chain with $S = \{1, 2, 3, 4, 5, 6, 7\}$, and transition probabilities

$$(p_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 4/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/5 & 2/5 & 2/5 & 0 & 0 \\ 1/10 & 0 & 0 & 0 & 7/10 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) **[7]** Which states are recurrent and which are transient?

(b) **[7]** Compute f_{i1} for each $i \in S$. (Hint: leave f_{41} until last.)

[END; total points = 60]