STA447/2006 Midterm #2, March 21, 2019

(135 minutes; 9 questions; 4 pages; total points = 60)

[SOLUTIONS]

1. [5] Let $S = \{1, 2, 3, 4\}$, with $\pi_1 = 1/8$, $\pi_2 = 3/8$, and $\pi_3 = \pi_4 = 1/4$. Find (with proof) transition probabilities $\{p_{ij}\}_{i,j \in S}$ for a Markov chain on S, such that $p_{ij} = 0$ whenever $|i - j| \geq 2$, and $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$.

Solution. *The Metropolis algorithm says we can let* $p_{i,i+1} = \frac{1}{2}\min[1, \frac{\pi_{i+1}}{\pi_i}]$ *and* $p_{i,i-1} = \frac{1}{2}\min[1, \frac{\pi_{i-1}}{\pi_i}]$ *and* $p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}$. *Thus,* $p_{1,2} = p_{3,4} = 1/2$, $p_{2,3} = \frac{1}{2}[(1/4)/(3/8)] = 1/3$, *and* $p_{3,2} = p_{4,3} = 1/2$, $p_{2,1} = \frac{1}{2}[(1/8)/(3/8)] = 1/6$, *and then* $p_{1,1} = 1 - (1/2) = 1/2$, $p_{2,2} = 1 - (1/3) - (1/6) = 1/2$, $p_{3,3} = 1 - (1/2) - (1/2) = 0$, *and* $p_{4,4} = 1 - (1/2) = 1/2$. *That is,*

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/6 & 1/2 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Then $p_{ij} = 0$ *whenever* $|i - j| \geq 2$. *And* P *is reversible with respect to* π *by construction, so* π *is stationary. Also the chain is irreducible since it is possible to go* $1 \to 2 \to 3 \to 4$ *and* $4 \to 3 \to 2 \to 1$. *And the chain is aperiodic since e.g.* $p_{1,1} > 0$. *So, by the Markov Chain Convergence Theorem,* $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ *for all* $i, j \in S$.

2. [5] Consider the Markov chain with state space $S = \{1, 2, 3, 4\}$, $\nu_3 = 1$, and transition probabilities specified by $p_{11} = p_{22} = 1$, $p_{31} = p_{32} = p_{33} = p_{34} = 1/4$, and $p_{42} = p_{43} = p_{44} = 1/3$. Compute $\mathbf{P}_3(T_1 < T_2)$. [Hint: Don't forget how we solved Gambler's Ruin.]

Solution. *Let* $s(a) = \mathbf{P}_a(T_1 < T_2)$. *Then* $s(1) = 1$ *since we've already reached 1, and* $s(2) = 0$ *since we have already reached 2. Also, by conditioning on the first step (just like for solving Gambler's Ruin), we have that for* $a = 3$ *or* 4, $s(a) = \sum_j p_{aj} s(j)$. *So, setting* $a = 3$, $s(3) = (1/4)s(1) + (1/4)s(2) + (1/4)s(3) + (1/4)s(4) = (1/4) + (1/4)s(3) + (1/4)s(4)$, *i.e.* $4s(3) = 1 + s(3) + s(4)$, *i.e.* $3s(3) - 1 = s(4)$. *Also, setting* $a = 4$, $s(4) = (1/3)s(2) + (1/3)s(3) + (1/3)s(4) = (1/3)s(3) + (1/3)s(4)$, *i.e.* $(2/3)s(4) = (1/3)s(3)$, *i.e.* $s(4) = (1/2)s(3)$. *Hence,* $3s(3) - 1 = (1/2)s(3)$, *i.e.* $(5/2)s(3) = 1$, *so* $s(3) = 2/5$, *i.e.* $\mathbf{P}_3(T_1 < T_2) = 2/5$.

3. Consider a graph with vertex set $V = \{1, 2, 3, 4\}$, and edge weights $w(1, 2) = w(2, 1) = 2$, $w(1, 3) = w(3, 1) = 3$, $w(1, 4) = w(4, 1) = 4$, and $w(u, v) = 0$ otherwise. Let $\{X_n\}$ be random walk on this graph, with $X_0 = 1$.

(a) [2] Compute (with explanation) $\mathbf{P}(X_1 = 4)$.

Solution. *Since* $X_0 = 1$, $\mathbf{P}(X_1 = 4) = p_{14} = w(1, 4)/d(4) = w(1, 4)/\sum_j w(1, j) = 4/(2 + 3 + 4) = 4/9$.

(b) [3] Compute (with explanation) $\mathbf{P}(X_3 = 4)$.

Solution. *Here* $\mathbf{P}(X_3 = 4) = p_{14}^{(3)} = \sum_{j,k} p_{1j} p_{jk} p_{k4} = \sum_j p_{1j} p_{j1} p_{14} = (2/9)(1)(4/9) + (3/9)(1)(4/9) + (4/9)(1)(4/9) = 4/9$.

(c) [**4**] For each of **(i)** $\lim_{n\to\infty} \mathbf{P}(X_n = 4)$, and **(ii)** $\lim_{n\to\infty} \frac{1}{2}[\mathbf{P}(X_n = 4) + \mathbf{P}(X_{n+1} = 4)]$, determine whether or not the limit exists, and if yes then what it equals.

Solution. *This graph is bipartite (with subsets* $\{1\}$ *and* $\{2,3,4\}$*), so* $\mathbf{P}(X_n = 4) = 0$ *for* n *even, while* $\mathbf{P}(X_n = 4) > 0$ *(and converges to* $2\,d(4)/Z$*)) for* n *odd. So,* $\lim_{n\to\infty} \mathbf{P}(X_n = 4)$ *does* <u>*not*</u> *exist. However, since the graph is connected, and* $Z = \sum_u d(u) = 9+2+3+4 = 18 < \infty$*, by Graph Average Convergence we have that* $\lim_{n\to\infty} \frac{1}{2}[\mathbf{P}(X_n = 4) + \mathbf{P}(X_{n+1} = 4)] = d(4)/Z = 4/18 = 2/9$.

4. [**5**] Suppose we repeatedly roll a fair six-sided die (which is equally likely to show 1, 2, 3, 4, 5, or 6). Let τ be the number of rolls until we see 5 twice in a row, i.e. until the pattern "55" first appears. Let $z = \mathbf{E}(\tau)$. Compute z.

Solution. *Let* X_n *be the amount of the pattern "55" that we have achieved after the* n^{th} *roll (starting over as soon as we complete it). Then* $\{X_n\}$ *is a Markov chain on* $S = \{0, 1, 2\}$*, with transitions* $p_{00} = 5/6$*,* $p_{01} = 1/6$*,* $p_{10} = 5/6$*,* $p_{12} = 1/6$*,* $p_{20} = 5/6$*, and* $p_{21} = 1/6$*. Its stationary distribution* π *must satisfy that* $\pi P = \pi$*, i.e.* $\pi_0 p_{0j} + \pi_1 p_{1j} + \pi_2 p_{2j} = \pi_j$ *for all* $j \in S$*. Setting* $j = 0$ *gives* $\pi_0(5/6) + \pi_1(5/6) + \pi_2(5/6) = \pi_0$*, and since* $\pi_0 + \pi_1 + \pi_2 = 1$ *this means that* $\pi_0 = 5/6$*. Setting* $j = 1$ *gives* $\pi_0(1/6) + \pi_2(1/6) = \pi_1$*, i.e.* $\pi_1 = (5/36) + (1/6)\pi_2$*. Setting* $j = 2$ *gives* $\pi_1(1/6) = \pi_2$*, i.e.* $\pi_1 = 6\pi_2$*. Thus,* $6\pi_2 = (5/36) + (1/6)\pi_2$*, whence* $(35/6)\pi_2 = 5/36$*, so* $\pi_2 = (5/36)/(35/6) = (5/6)/35 = 1/(6 \times 7) = 1/42$*. But* z *is the expected time to go from 0 to 2, or equivalently the mean recurrence time of the state 2. Hence, by the Recurrence Time Theorem,* $z = 1/\pi_2 = 42$.

5. [**4**] In the previous question, let X be the sum of all the numbers up to but <u>not</u> including the first "55", and let Y be the sum of all the numbers up to <u>and</u> including the first "55". Compute $\mathbf{E}(X)$ and $\mathbf{E}(Y)$. [Note: If you could not solve the previous question, then you may leave your answers to this question in terms of the unknown value z.]

Solution. *Here* τ *is a stopping time with finite mean. And,* Y *is a sum of i.i.d. dice rolls up to time* τ*, each with mean 3.5. Hence, by Wald's Theorem,* $\mathbf{E}(Y) = (3.5)\mathbf{E}(\tau) = (3.5)z = (3.5)(42) = 147$*. Now,* $\tau - 2$ *is* <u>*not*</u> *a stopping time (since it looks into the future), so we* <u>*cannot*</u> *use Wald's Theorem for* X*. But we always have* $X = Y - 10$ *whence* $\mathbf{E}(X) = \mathbf{E}(Y) - 10 = (3.5)z - 10 = 147 - 10 = 137$.

6. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4\}$, with $X_0 = 3$, and with transition probabilities $p_{11} = p_{44} = 1$, $p_{21} = 1/4$, $p_{34} = 1/5$, and $p_{24} = p_{31} = p_{12} = p_{13} = p_{14} = p_{41} = p_{42} = p_{43} = 0$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } 4\}$, and let $U = T - 1$.

(a) [**4**] Find valid values of p_{22}, p_{23}, p_{32}, and p_{33}, which make $\{X_n\}$ a <u>martingale</u>.

Solution. *We need* $\sum_j j\, p_{2j} = 2$*, i.e.* $p_{21}(1) + p_{22}(2) + p_{23}(3) = 2$*, whence* $p_{23} = p_{21} = 1/4$*. Then* $p_{22} = 1-(1/4)-(1/4) = 1/2$*. And, we need* $\sum_j j\, p_{3j} = 3$*, i.e.* $p_{32}(2)+p_{33}(3)+p_{34}(4) = 3$*, whence* $p_{32} = p_{34} = 1/5$*. Then* $p_{33} = 1 - (1/5) - (1/5) = 3/5$.

(b) [**2**] For the values found in part (a), compute $\mathbf{E}(X_T)$.

Solution. *Clearly the chain is bounded up to time* T*, indeed we always have* $|X_n| \leq 4$*. Hence, by the Optional Stopping Corollary,* $\mathbf{E}(X_T) = \mathbf{E}(X_0) = 3$.

(c) **[3]** For the values found in part (a), compute $p = \mathbf{P}(X_T = 4)$.

Solution. *We must have* $X_T = 1$ *or 4, so* $\mathbf{P}(X_T = 1) = 1-p$, *and* $\mathbf{E}(X_T) = p(4)+(1-p)(1) = 1+3p$. *Hence, by part (b),* $3 = 1+3p$, *so* $p = 2/3$, *i.e.* $\mathbf{P}(X_T = 4) = 2/3$.

(d) **[3]** For the values found in part (a), compute $\mathbf{E}(X_U)$.

Solution. *Here* U *is* not *a stopping time, so we* cannot *apply the Optional Stopping Theorem. However, if* $X_T = 1$ *then we must have* $X_U = 2$, *while if* $X_T = 4$ *then we must have* $X_U = 3$. *Hence,* $\mathbf{E}(X_U) = \sum_\ell \ell\,\mathbf{P}(X_U = \ell) = (2)\,\mathbf{P}(X_U = 2) + (3)\,\mathbf{P}(X_U = 3) = (2)\,\mathbf{P}(X_T = 1) + (3)\,\mathbf{P}(X_T = 4) = (2)(1/3) + (3)(2/3) = 8/3$.

7. Consider a Markov chain $\{X_n\}$ with state space $S = \{0, 1, 2, 3, \ldots\}$, with $p_{0,0} = 1$, and $p_{i,0} = p_{i,2i} = 1/2$ for all $i \geq 1$, and with $X_0 = 5$. Let $T = \inf\{n \geq 1 : X_n = 0\}$.

(a) **[2]** Determine whether or not $\{X_n\}$ is a martingale.

Solution. *Yes. For each* $i \in S$, *we have* $\sum_j j\,p_{ij} = (0)(1/2) + (2i)(1/2) = i$. *Also,* $|X_n| \leq 5(2^n)$ *so* $\mathbf{E}|X_n| < \infty$. *Hence,* $\{X_n\}$ *is a martingale.*

(b) **[2]** Determine whether or not $\mathbf{E}(X_n) = 5$ for each fixed $n \in \mathbf{N}$.

Solution. *Yes. Since* $\{X_n\}$ *is a martingale, therefore* $\mathbf{E}(X_n) = \mathbf{E}(X_0) = 5$ *for each fixed* $n \in \mathbf{N}$.

(c) **[2]** Determine whether or not $\mathbf{P}(T < \infty) = 1$.

Solution. *Yes. Here we have probability 1/2 of moving to 0 at each step, so* $\mathbf{P}(T \geq k) = (1/2)^k$ *which* $\to 0$ *as* $k \to \infty$, *i.e.* $\mathbf{P}(T = \infty) = 0$, *so* $\mathbf{P}(T < \infty) = 1$. *(Aside: This also means that* $X_n \to 0$, *i.e. that* $\{X_n\}$ *converges with probability 1, as it must do by the Martingale Convergence Theorem.)*

(d) **[2]** Determine whether or not $\mathbf{E}(X_T) = 5$.

Solution. *No. Here we always have* $X_T = 0$, *whence* $\mathbf{E}(X_T) = 0 \neq 5$.

8. Let $\{B_t\}_{t \geq 0}$ be standard Brownian motion, and let $\tau = \inf\{t > 0 : B_t = -2 \text{ or } 3\}$.

(a) **[3]** Compute $\mathbf{E}[(2 + B_2 + B_3)^2]$.

Solution. *Here* $\mathbf{E}[(2+B_2+B_3)^2] = \mathbf{E}[4 + B_2^2 + B_3^2 + 4B_2 + 4B_3 + 2B_2B_3] = 4 + \mathrm{Var}(B_2) + \mathrm{Var}(B_3) + 4\mathbf{E}(B_2) + 4\mathbf{E}(B_3) + 2\mathrm{Cov}(B_2, B_3) = 4 + 2 + 3 + 4(0) + 4(0) + 2\min[2,3] = 4 + 2 + 3 + 0 + 0 + 4 = 13$.

(b) **[3]** Compute $p = \mathbf{P}[B_\tau = 3]$.

Solution. *Here* τ *is a stopping time, and* $\{B_t\}$ *is bounded (between* -2 *and* 3*) up to time* τ. *So, by the Optional Stopping Corollary,* $\mathbf{E}(B_\tau) = \mathbf{E}(B_0) = 0$, *i.e.* $p(3) + (1-p)(-2) = 0$, *i.e.* $5p - 2 = 0$, *so* $p = 2/5$.

9. Suppose cars arrive according to a Poisson process with rate $\lambda = 3$ cars per minute, and each car is independently either Blue with probability 1/2, or Green with probability 1/3, or Red with probability 1/6.

(a) **[3]** Let S be the arrival time of the <u>first</u> car that arrives <u>after</u> at least 5 minutes (so we must have $S > 5$). Compute (with explanation) the expected value $\mathbf{E}(S)$.

Solution. *The interarrival times of a Poisson Process with rate* λ *are Exponential*(λ). *By the memoryless property of the Exponential distribution, the time to the next arrival after 5 minutes has the same distribution. So,* $S = 5 + U$, *where* $U \sim \text{Exponential}(3)$. *Hence,* $\mathbf{E}(S) = 5 + \mathbf{E}(U) = 5 + (1/\lambda) = 5 + (1/3) = 16/3$.

(b) **[3]** Compute (with explanation) the probability that, in the first 2 minutes, exactly 2 Blue and 1 Green cars arrive.

Solution. *By Poisson Thinning, the number of Blue cars is a Poisson Process with rate* $\lambda_1 = \lambda(1/2) = 3/2$, *and the number of Green cars are a Poisson Process with rate* $\lambda_2 = \lambda(1/3) = 1$, *and the two processes are* <u>*independent*</u>. *Hence, the probability that exactly 2 Blue and 1 Green cars arrive in the first 2 minutes is equal to*

$$\left(e^{-2\lambda_1}\frac{[2\lambda_1]^2}{2!}\right)\left(e^{-2\lambda_2}\frac{[2\lambda_2]^1}{1!}\right) \;=\; \left(e^{-2(3/2)}\frac{[2(3/2)]^2}{2}\right)\left(e^{-2(1)}\frac{[2(1)]^1}{1}\right) \;=\; 9\,e^{-5}\,.$$

[END OF EXAMINATION; total points = 60]