STA447/2006 Midterm #1, February 6, 2020

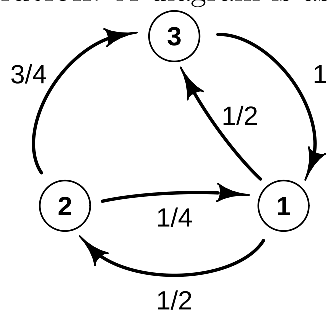
(135 minutes; 6 questions; 4 pages; total points = 50)

[SOLUTIONS]

1. Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{12} = 1/2$, $p_{13} = 1/2$, $p_{21} = 1/4$, $p_{23} = 3/4$, and $p_{31} = 1$, otherwise $p_{ij} = 0$.

(a) [**2**] Draw a diagram of this Markov chain.

Solution. *A diagram is as follows:*



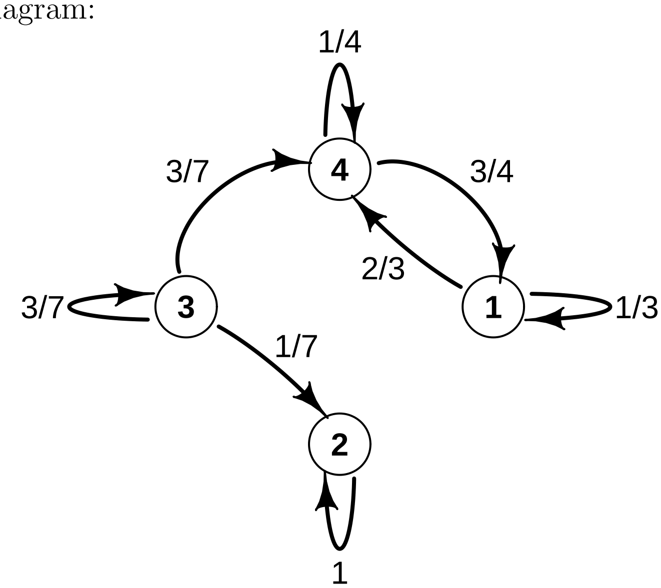
(b) [**2**] Compute $p_{11}^{(2)}$.

Solution. $p_{11}^{(2)} = \sum_{j \in S} p_{1j} p_{j1} = p_{11}p_{11} + p_{12}p_{21} + p_{13}p_{31} = (0)(0) + (1/2)(1/4) + (1/2)(1) = (1/8) + (1/2) = 5/8.$

(c) [**3**] Determine (with explanation) whether or not $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$.

Solution. *Yes it does. The chain is irreducible since e.g. it can go* $1 \to 2 \to 3 \to 1 \to 2$. *And* $|S| = 3$ *which is finite. So, by the Finite Space Theorem,* $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ *for all* i *and* j*, including when* $i = 1$ *and* $j = 2$.

2. Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition probabilities as in the following diagram:



(a) **[3]** Compute f_{41}.

Solution. *Solution #1: $C = \{1,4\}$ is a closed irreducible finite subset, so applying the Closed Subset Note to the Finite State Space Theorem and Recurrence Equivalences Theorem, we must have $f_{ij} = 1$ for all $i, j \in C$, including when $i = 4$ and $j = 1$.*

Solution #2: Starting from state 4, the only way the chain could avoid hitting state 1 is if it stays at state 4 forever, which has probability $\lim_{n\to\infty} \mathbf{P}_4[X_1 = X_2 = \ldots = X_n] = \lim_{n\to\infty}(1/4)^n = 0$. Hence, the chain must eventually hit state 1, so $f_{41} = 1$.

Solution #3: Starting from state 4, the probability that the chain first hits state 1 at time τ is equal to $(p_{44})^{\tau-1}(p_{41}) = (1/4)^{\tau-1}(3/4)$. So, $f_{41} = \sum_{\tau=1}^{\infty}(1/4)^{\tau-1}(3/4) = (3/4)\frac{1}{1-(1/4)} = \frac{3/4}{3/4} = 1$.

(b) **[3]** Compute f_{31}.

Solution. *Solution #1: By the f-Expansion, since $2 \not\to 1$ so $f_{21} = 0$, $f_{31} = p_{31} + \sum_{k\neq 1} p_{3k} f_{k1} = p_{31} + p_{32}f_{21} + p_{33}f_{31} + p_{34}f_{41} = 0 + (1/7)(0) + (3/7)f_{31} + (3/7)(1) = (3/7)f_{31} + (3/7)$. Hence, $(4/7)f_{31} = 3/7$, so $f_{31} = 3/4$.*

Solution #2: Starting from state 3, when the chain finally leaves state 3 then it must move to either state 4 (after which it must eventually hit state 1 since $f_{41} = 1$), or to state 2 (after which it can never hit state 3). So, $f_{31} = \mathbf{P}_3[X_1 = 4 \,|\, X_1 \neq 3] = \frac{p_{34}}{p_{34}+p_{32}} = \frac{3/7}{(3/7)+(1/7)} = 3/4$.

Solution #3: Starting from state 3, the probability that the chain first hits state 4 at time τ is equal to $(p_{33})^{\tau-1}(p_{34}) = (3/7)^{\tau-1}(3/7)$. So, $f_{34} = \sum_{\tau=1}^{\infty}(3/7)^{\tau-1}(3/7) = (3/7)\frac{1}{1-(3/7)} = \frac{3/7}{4/7} = 3/4$. Then, since the only way to get from state 3 to state 1 is to first visit state 4, we must have $f_{31} = f_{34}f_{41} = (3/4)(1) = 3/4$.

(c) **[4]** Compute $\sum_{n=1}^{\infty} p_{33}^{(n)}$, and determine if state 3 is recurrent or transient.

Solution. *Once the chain leaves state 3 it can never return, so $p_{33}^{(n)} = (p_{33})^n = (3/7)^n$. Hence, $\sum_{n=1}^{\infty} p_{33}^{(n)} = \sum_{n=1}^{\infty}(3/7)^n = \frac{3/7}{1-(3/7)} = \frac{3/7}{4/7} = 3/4$. Then, since $3/4 < \infty$, state 3 must be transient by the Recurrent State Theorem.*

3. For each of the following sets of conditions, either provide (with explanation) an example of a state space S (which contains states 1 and 2, but might also contain other states too), and Markov chain transition probabilities $\{p_{ij}\}_{i,j\in S}$, such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) **[3]** The chain is irreducible, and $\sum_{n=1}^{\infty} p_{12}^{(n)} < \infty$, and $f_{12} = 1$.

Solution. *Yes, possible. For example, take simple random walk with $p = 0.6$ (or any $1/2 < p < 1$). Then the chain is irreducible since it is srw, and transient since $p \neq 1/2$. Hence, $\sum_{n=1}^{\infty} p_{12}^{(n)} < \infty$ by the Transience Equivalences Theorem. However, $f_{12} = 1$ by Proposition 1.6.17 since $p > 1/2$.*

(b) **[3]** $\sum_{n=1}^{\infty} p_{11}^{(n)} = \infty$, and $p_{21} > 0$, but $f_{21} < 1$.

Solution. *Yes, possible. For example, let $S = \{1,2,3\}$, and $p_{11} = p_{33} = 1$, and $p_{21} = p_{23} = 1/2$. Then $\sum_{n=1}^{\infty} p_{11}^{(n)} = \sum_{n=1}^{\infty}(1) = \infty$. Also, $p_{21} = 1/2 > 0$. However, f_{21} is the probability from 2 that the chain will eventually hit 1, which it can only do on its first step or never, so $f_{21} = p_{21} = 1/2 < 1$.*

(c) [3] For all $n \in \mathbf{N}$, $p_{12}^{(n)} \geq 1/3$ and $p_{21}^{(n)} \geq 1/5$, and state 2 is transient.

Solution. *Impossible. If $p_{12}^{(n)} \geq 1/3$ and $p_{21}^{(n)} \geq 1/5$, then by the Chapman-Kolmogorov Inequality, $p_{22}^{(2n)} \geq p_{21}^{(n)} p_{12}^{(n)} \geq (1/5)(1/3) = 1/15$ for all $n \in \mathbf{N}$, so $\sum_{n=1}^{\infty} p_{22}^{(n)} \geq \sum_{n=1}^{\infty} p_{22}^{(2n)} \geq \sum_{n=1}^{\infty}(1/15) = (1/15)(\infty) = \infty$. Hence, by the Recurrent State Theorem, state 2 must be recurrent, not transient.*

4. Suppose a Markov chain has distinct states $i, j \in S$, with i recurrent, and j transient. (Of course, S might also contain other states too.)

(a) [3] Show by example that it is possible that $j \to i$.

Solution. *For example, let $S = \{1, 2, 3\}$, and $p_{11} = p_{33} = 1$, and $p_{21} = p_{23} = 1/2$. Then 1 is recurrent since it always stays there, while 2 is transient since it always leaves there and never returns, and $2 \to 1$ since $p_{21} > 0$.*

(b) [5] Prove that it is impossible that $i \to j$.

Solution. *Suppose to the contrary that $i \to j$. Then $f_{ij} > 0$. Also, since i is recurrent, $f_{ii} = 1$. Then, the f-Lemma says we must have $f_{ji} = 1 > 0$. So, $i \leftrightarrow j$. Hence, by the Sum Corollary, i is recurrent iff j is recurrent. This contradicts the fact that i is recurrent but j is not. Hence, it is impossible that $i \to j$.*

5. Consider a Markov chain having states $i, j \in S$, such that $\mathbf{P}_i[N(j) = \infty] = 1/3$. (Of course, S might also contain other states too.)

(a) [3] Prove that any such chain cannot have $i = j$.

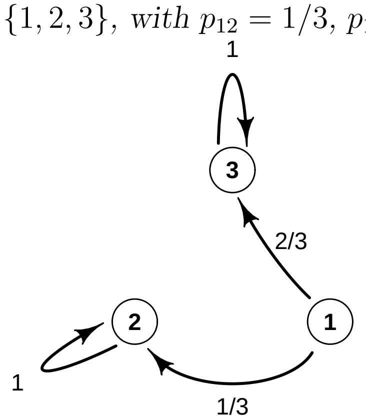
Solution. *If $i = j$, then by the Recurrent State Theorem, either $\mathbf{P}_i[N(i) = \infty] = 1$ (if state i is recurrent), or $\mathbf{P}_i[N(i) = \infty] = 0$ (if state i is transient), but we can never have $\mathbf{P}_i[N(i) = \infty] = 1/3$.*

(b) [3] Prove that any such chain cannot be irreducible.

Solution. *If the chain is irreducible, then by the Infinite Returns Lemma (or the Recurrence Equivalences Theorem plus the Transience Equivalences Theorem), we must have either $\mathbf{P}_i[N(i) = \infty] = 1$ (if the chain is recurrent), or $\mathbf{P}_i[N(i) = \infty] = 0$ (if the chain is transient), but again we can never have $\mathbf{P}_i[N(i) = \infty] = 1/3$.*

(c) [3] Provide (with explanation) a valid example of such a chain.

Solution. *For example, let $S = \{1, 2, 3\}$, with $p_{12} = 1/3$, $p_{13} = 2/3$, and $p_{22} = p_{33} = 1$:*



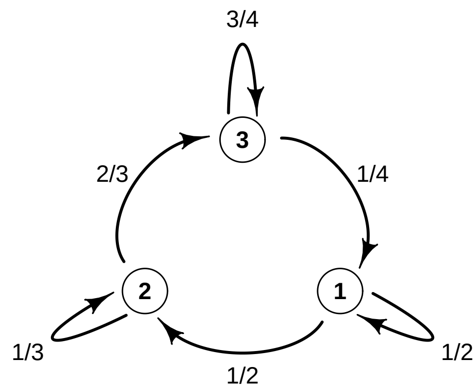
Then from state 1, on the first step the chain will either jump to state 2 (after which it will stay there forever, and hit state 2 an infinite number of times), or jump to state 3 (after which it will stay there forever, and never hit state 2 at all). Hence, $\mathbf{P}_1[N(2) = \infty] = \mathbf{P}_1[X_1 = 2] = p_{12} = 1/3$*, as desired (with* $i = 1$ *and* $j = 2$*).*

6. [7] Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 2/3 \\ 1/4 & 0 & 3/4 \end{pmatrix}$$

Either compute $\lim_{n\to\infty} p_{12}^{(n)}$, or prove that the limit does not exist.

Solution. *A diagram of the Markov chain is as follows:*



The chain is irreducible since e.g. it can go $1 \to 2 \to 3 \to 1 \to 2$.

And, it is aperiodic since e.g. $p_{11} > 0$.

If $\{pi_i\}_{i\in S}$ *is a stationary distribution, then it must satisfy* $\sum_{i\in S} \pi_i p_{ij} = \pi_j$ *for all* $j \in S$.

$j = 1$: $\pi_1(1/2) + \pi_2(0) + \pi_3(1/4) = \pi_1$, *so* $\pi_3 = 2\pi_1$.

$j = 2$: $\pi_1(1/2) + \pi_2(1/3) + \pi_3(0) = \pi_2$, *so* $\pi_2 = (3/4)\pi_1$.

We need $\pi_1 + \pi_2 + \pi_3 = 1$, *i.e.* $\pi_1 + (3/4)\pi_1 + 2\pi_1 = 1$, *i.e.* $(15/4)\pi_1 = 1$, *so* $\pi_1 = 4/15$.

Then $\pi_2 = (3/4)\pi_1 = (3/4)(4/15) = 3/15 = 1/5$, *and* $\pi_3 = 2\pi_1 = (2)(4/15) = 8/15$, *so* $\pi = (4/15, 1/5, 8/15)$.

As a check, $j = 3$: $\pi_1(0)+\pi_2(2/3)+\pi_3(3/4) = \pi_3$, *so* $\pi_3 = (8/3)\pi_2$, *which is correct since* $8/15 = (8/3)(1/5)$.

We conclude that $\pi = (4/15, 1/5, 8/15)$ *is a valid stationary distribution for this chain.*

Then, since the chain is irreducible and aperiodic and has a stationary distribution, by the Markov Chain Convergence Theorem we must have $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ *for all* $i, j \in S$.

Choosing $i = 1$ *and* $j = 2$, *we must have* $\lim_{n\to\infty} p_{12}^{(n)} = \pi_2 = 1/5$ *(and yes the limit exists).*

[END OF EXAMINATION; total points = 50]